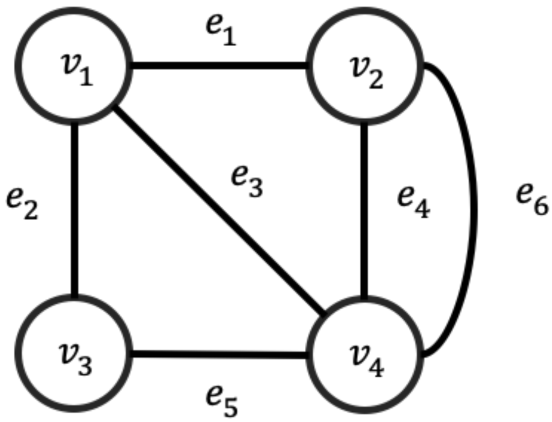
Exercise: List the degree of each vertex in the following graphs.

Verify $\sum_{\{v \in V\}} deg(v) = 2m$

Solution:



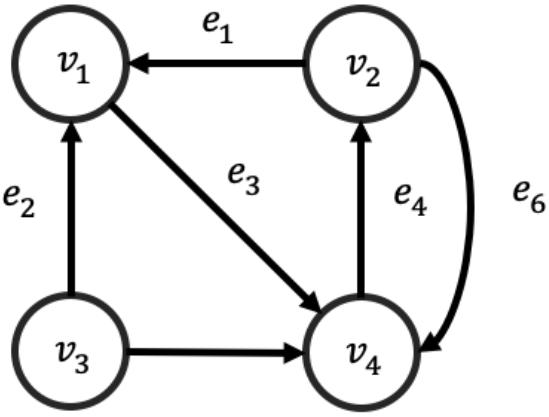
We have $m = |E| = 6$, hence $2m = 12$.

$$\sum_{\{v \in V\}} deg(v) = deg(v_1) + deg(v_2) + deg(v_3) + deg(v_4)$$
$$= 3 + 3 + 2 + 4 = 12$$

Exercise: List the degree of each vertex in the following graphs. Verify

$$\sum_{\{v \in V\}} deg^{-}(v) = \sum_{\{v \in V\}} deg^{+}(v) = m,$$

Definition: *In degree of a vertex v $\left(deg^{-}(v)\right)$ is the number of edges with v as their terminal vertex.*
Out degree of a vertex v $\left(deg^{+}(v)\right)$ is the number edges with v as their initial vertex.



Solution: We have $m = |E| = 6$ edges

$$\sum_{\{v \in V\}} deg^{-}(v) = deg^{-}(v_1) + deg^{-}(v_2) + deg^{-}(v_3) + deg^{-}(v_4))$$
$$= 1 + 2 + 2 + 1 = 6$$

$$\sum_{\{v \in V\}} deg^{+}(v) = deg^{+}(v_1) + deg^{+}(v_2) + deg^{+}(v_3) + deg^{+}(v_4))$$
$$= 2 + 1 + 0 + 3 = 6$$

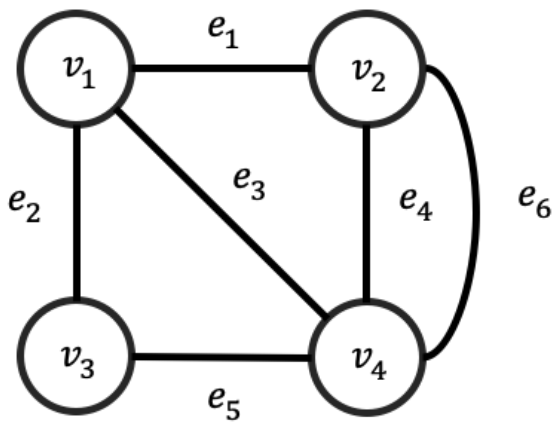
So verified!

Exercise: How many edges are there in an undirected graph with 10 vertices each of degree ?

Solution:

$$\begin{aligned} 2m &= \sum_{\{v \in V\}} deg(v) \\ &= 10 \cdot 6 \\ &= 60 \\ \Rightarrow m &= 60 \end{aligned}$$

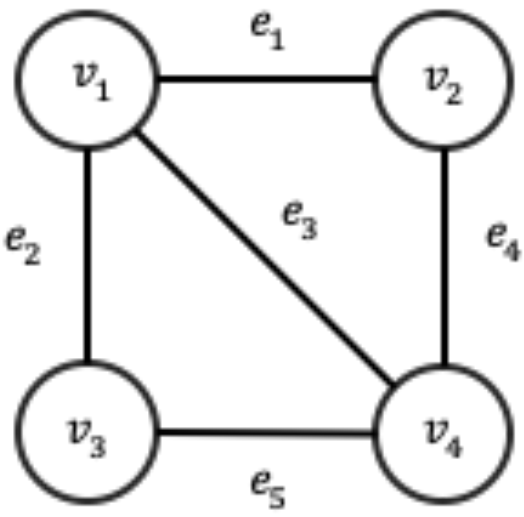
Exercise: Write down the adjacency matrix of of the graph G.



Solution: Let $n = |V|$ be the number of vertices. An adjacency matrix A of G with is an $n \times n$ matrix with elements $a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of G. In this example every combination of the vertices are edges in G except for $\{v_2, v_3\}$, thus $a_{23} = a_{32} = 0$, thus

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

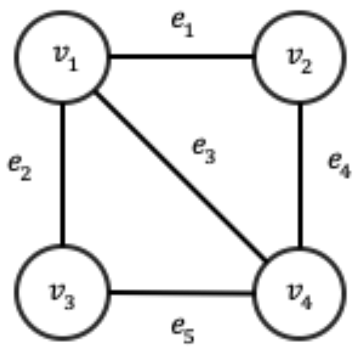
Exercise: Write down the incidence matrix of the graph G

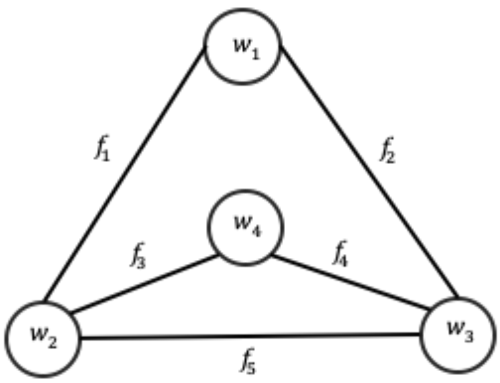


The incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $M = [m_{ij}]$ where $m_{ij} = 1$ if edge e_j is incident with v_i thus we have

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Exercise: Prove that the following two graphs are isomorphic





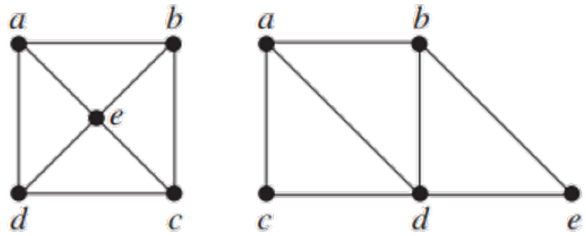
Definition:

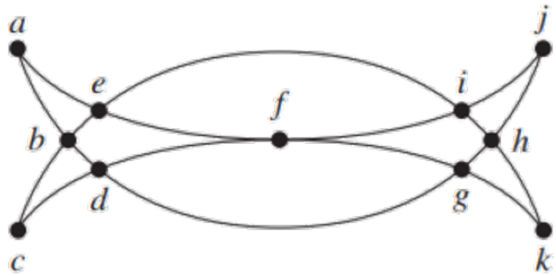
The simple graphs $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$, *are said to be isomorphic if there exists a on – to – one and onto function* f *from* V_1 *to* V_2 *with the property that a and b are adjacent in* G_1 *if and only if* $f(a)$ *and* $f(b)$ *are adjacent in* G_2, *for all a and b in* V_1. *Such a function* f *is called an isomorphism.*
Two simple graphs that are not ismorphic is called nonisomorphic.

Both graphs contain four vertices and five edges. Two vertices have $deg = 2$ and two vertices have $deg = 3$ in both graphs. It is also easy to see that all subgraphs of are isomorphic. Because they agree with respect to these invariants, it is reasonable to try find an isomorphism f.
Because v_1 and v_4 are connected to two vertices with $deg = 2$ and v_2 and v_3 are connected to two vertices of $deg = 3$ we must map these to verticies with the same proper So $f(v_1) = w_2$ and $f(v_4) = w_3$ as v_1, v_4 and w_2,w_3 have the same properties. Thus $f(v_2) = w_1,\ f(v_3) = w_4$
To se wheter f preserves edges, we examine the adjacency matrices.

$$\begin{array}{cccc|c} v_1 & v_2 & v_3 & v_4 & \\ \hline 0 & 1 & 1 & 1 & v_1 \\ 1 & 0 & 0 & 1 & v_2 \\ 1 & 0 & 0 & 1 & v_3 \\ 1 & 1 & 1 & 0 & v_4 \end{array} \qquad \begin{array}{cccc|c} w_2 & w_1 & w_4 & w_3 & \\ \hline 0 & 1 & 1 & 1 & w_2 \\ 1 & 0 & 0 & 1 & w_1 \\ 1 & 0 & 0 & 1 & w_4 \\ 1 & 1 & 1 & 0 & w_3 \end{array}$$

Exercise: Do the following graph contain an Euler path respectively? What about an Euler circuit?





Reminde yourself

Definition:

An Euler circuit is a graph G that is a simple circuit containing every edge of G.
An Euler
path in G is a simple path containing every edge of G

Nessesary conditions for existence of Euler circuit/path:

Theorem:

A connected multigraph with atleast two vertices has an Euler circuit if and only if each of its vertices has an even deggree

A connected multigraph has an Euler path but not an Euler circuit if and only if it has two vertices of odd degree.

The first (upper left corner) graph does not contain an Euler circuit since four out of five vertex have odd degree. neither does it contain a Euler path.

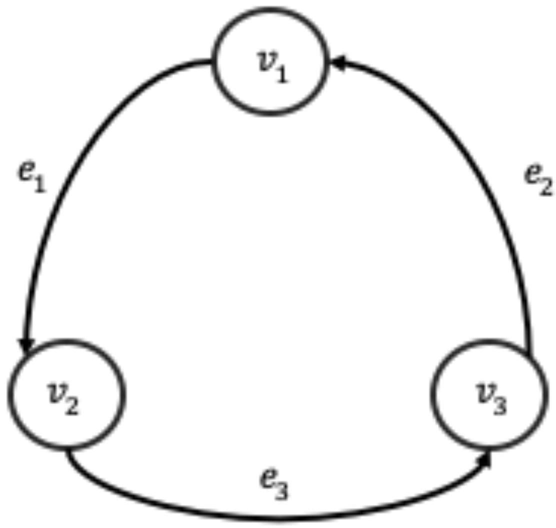
The second upper right corner) graph has an Euler circuit since $deg(a) = deg(b) = 3$ and $deg(c) \ = 2, \ deg(d) = 4, \ deg(e) \ = \ 2$ thus we have exactly two verticies with odd degre, therefore it has an Euler path but not an Euler circuit.

In Mohammad Scimitars each vertex has $deg = 2$ so it is a Euler circuit. The graph howev does not have any vertex of odd degree so there is no Euler path.

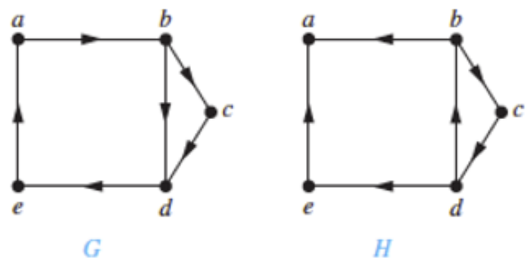
Exercise: Is this graph strongly / weakly connected?

Definition:
$A\ graph\ is\ called\ strongly\ connected\ if\ there\ is\ a\ path\ from\ a\ to\ b\ and\ from\ b\ to\ a\ f$
$or\ all\ vertives\ \{a, b\}$

$A\ graph\ is\ called\ weakly\ connected\ if\ there\ is\ a\ path\ between\ every\ two\ vertices$
$in\ the\ underlying\ undirected\ graph.$



Solution:
It is clearly also strongly connected because we can go from v_1 to v_3 by the road $v_1 \to v_2 \to v_3$ and we can go from v_3 to v_1 by $v_3 \to v_1$



G is clearly strongly connected and therefore also weakly connected

H is weak. If undirected it is clearly connected! If it is directed then you can go $b \to c \to d \to e$ but there is no way to go from e to b as the only walk you can do from e is $e \to a$.