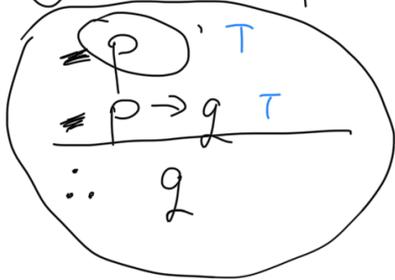


Proofs

Rules of inference

① Modus ponens



① "Today is Tuesday" P

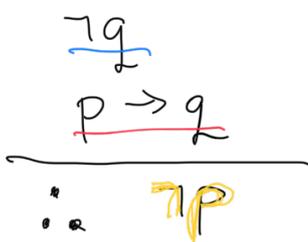
① "If today is Tuesday, Bob is working" $P \rightarrow Q$

"Bob is working" must be true.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

So, in mathematics:
if P holds (i.e., is true),
and $P \rightarrow Q$ holds, then
 Q must hold.

② Modus tollens



"not (Bob is working)" $\neg Q$

"if today is Tuesday, Bob is working" $P \rightarrow Q$

"not (today is Tuesday)" $\neg P$

P	Q	$\neg Q$	$P \rightarrow Q$
T	T	F	T
T	F	T	F
F	T	F	T
F	F	T	T

modus tollens via contraposition

reminder:

P Q	$P \rightarrow Q$	$\neg P$ $\neg Q$	$\neg Q \rightarrow \neg P$
T T	T	F F	T
T F	F	F T	F
F T	T	T F	T
F F	T	T T	T

contraposition: $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$

1. $P \rightarrow Q$ is given
2. $\neg Q$ is given
3. $\neg Q \rightarrow \neg P$ using the contraposition

$$\begin{array}{l} \neg Q \\ \neg Q \rightarrow \neg P \\ \hline \therefore \neg P \end{array}$$

Example using modus ponens

"for every integer x , if x is odd, then $x+1$ is even."

" x is odd" P

" $x+1$ is even" Q

"if x is odd, then $x+1$ is even" $P \rightarrow Q$

assume " x is odd" is true $(1, 3, 5, \dots)$

"being odd" means $x = 2 \cdot y + 1$

$$\begin{array}{l} 2 \cdot 0 + 1 = 1 \\ 2 \cdot 1 + 1 = 3 \end{array}$$

"let x be an integer, such that $x = 2y + 1$,"
"any" "odd"

Now: we need to show that $x+1$ is even.

"being even" means $w = 2 \cdot v$

$\Rightarrow x = 2y + 1$ we add +1 to both sides

$$x+1 = 2y+1+1$$

$$x+1 = 2y+2$$

$$x+1 = 2(\underbrace{y+1}_{=v})$$

$$x+1 = 2 \cdot v$$

even number

Theorem = statement, which has been proven on previously proven statements and axioms.

Axiom = generally accepted statements
 \rightarrow "every natural number n has exactly one successor $n+1$."

$$"1+1=2"$$

Proof = a clear and concise explanation that convinces another mathematician that a statement is true.

...

example 1

$(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})$ if x is odd and if y is odd then $x \cdot y$ is odd.

$$x = 2a + 1$$
$$y = 2b + 1$$

$$\begin{aligned} x \cdot y &= (2a+1)(2b+1) = \\ &= 4ab + 2a + 2b + 1 = \\ &= 2(2ab + a + b) + 1 \\ &= 2m + 1 \end{aligned}$$

by def. is an odd integer \square

$$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$$

" x is even if and only if $x+1$ is odd."

- a) if (x is even) then ($x+1$ is odd).
b) if ($x+1$ is odd) then (x is even).

(a) $x = 2v$ (even) $\Rightarrow x+1 = 2v+1$ (odd) ✓

(b) $x+1 = 2v+1$ $\Rightarrow x+1 = 2v+1$ ✓
 $x = 2v$
by def. even.

How to handle logical statements:

Statement

ways to prove it

⊗ p

- prove that p is true
- assume p is false, and derive a contradiction

$P \wedge Q$

- prove p and q ⊗

$P \vee Q$

- prove P
- prove Q
- assume $\neg P$ and deduce Q is true
- assume $\neg Q$ and deduce P is true

$P \rightarrow Q$

- assume p is true and deduce that q is true.

- assume $\neg Q$ and deduce $\neg P$.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$P \leftrightarrow Q$

- prove $(P \rightarrow Q)$ and $(Q \rightarrow P)$

- prove p and q

- prove $\neg p$ and $\neg q$

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F

F	T	F
F	T	T

$(\exists x \in S) P(x)$ - find an x in S such that $P(x)$

$(\forall x \in S) P(x)$ - prove that $P(x)$ holds for all x in S .

Example: for every integer x , if x is even, then x^2 is even.

Let x be any integer so that $x = 2y$.
 show: x^2 is even; $x^2 = 2t$

$$\begin{aligned} \rightarrow x &= 2y \\ x^2 &= 4y^2 = (2y)^2 \\ &= 2(2y^2) \\ &= 2t \end{aligned}$$

$x^2 = 2 \cdot t$ by definition is even.

Direct proof for statements $p: A \rightarrow B$.
 assume A and deduce B .

- shown a few times.

we assume A is true and then, using obvious, definitions and previously proven theorems to deduce that B must be true.

Example: Prove that if m, n are perfect squares, then $m \cdot n$ is also a perfect square.

A number x is a perfect square if

$$x = y^2$$

$9 = 3^2$	$3^2 \cdot 4^2 = 144$
$16 = 4^2$	$= 12^2$
$25 = 5^2$	

$$m = a^2, \quad n = b^2$$

$$\underline{m \cdot n} = a^2 \cdot b^2 = (a \cdot b)^2$$

therefore, $m \cdot n$ is also a perfect square. \square

Proof by contradiction: $p: A \rightarrow B$

reminder: $\neg(p \rightarrow q) \equiv p \wedge \neg q$

We assume A and $\neg B$ and derive a contradiction

Example: if x is rational and y is irrational, then $x+y$ is irrational.

x is rational: $x = \frac{p}{q}, \quad q \neq 0$

Assume the negation:

\checkmark x is rational: $x = \frac{p}{q}$

\rightarrow y is irrational

$x+y$ is not irrational $x+y = \frac{p'}{q'}, \quad q' \neq 0$

$$\frac{p}{q} + y = \frac{p'}{q'} \quad | - \frac{p}{q}$$

$$y = \frac{p'}{q'} - \frac{p}{q}$$

$$y = \frac{p' \cdot q - p \cdot q'}{q' \cdot q}$$

$$y = \frac{m}{n} \quad \text{by definition is a rational number!}$$

\star m, n are also $\in \mathbb{Z}$, as they are the result of operations between $p, q, p', q' \in \mathbb{Z}$

Contradicts our assumption

therefore the opposite must hold

Example: Prove that $\sqrt{2}$ is irrational.

Let's assume that $\sqrt{2}$ is rational.

$$\sqrt{2} = \frac{a}{b}, \quad b \neq 0, \quad a, b \in \mathbb{Z}$$

$\{1, 2, 3, \dots, 1\}$
 $\star a > 0$ also because of the same set \mathbb{Z}

- in addition, we demand $b > 0$

- now, we choose a, b such that they are the smallest number in our set from which we take b .

$$\frac{10}{20} = \frac{1}{2}$$

$$- a \quad |^2$$

$$\frac{2}{1}$$

$$\sqrt{2} = \frac{a}{b} \quad |$$

$$2 = \frac{a^2}{b^2} \quad | \cdot b^2$$

$$2b^2 = a^2$$

def. of an even number, so a itself is even (remember proof from before)

$$a = 2 \cdot k$$

$$2b^2 = (2k)^2$$

$$2b^2 = 4k^2 \quad | :2$$

$$b^2 = 2k^2$$

def. of an even number, so b itself is even

$$b = 2 \cdot m$$

So, there must be another number (m), which in turn must be smaller than b to fulfill the previous constraint.

- But, earlier we required that b is already the smallest number from our set.

So, we cannot $\sqrt{2}$ as $\frac{a}{b}$.

And, therefore, $\sqrt{2}$ cannot be rational, but must be irrational. \square

Example: Prove: if $3n+2$ is odd, then n is odd.

$$p: A \rightarrow B$$

$$\neg p: \neg(A \rightarrow B) \equiv A \wedge \neg B$$

$$A: 3n+2 \text{ is odd}$$

$$B: n \text{ is odd}$$

Proof by contradiction: $A \wedge \neg B$

$$A: 3n+2 \text{ is odd}$$

$$B: n \text{ is not odd} = \underline{n \text{ is even.}}$$

$$n = 2k$$

$3n+2$ is odd, use $n=2k$:

$$3(2k)+2 = 6k+2 = 2(3k+1)$$

$= 2m$ ^m
 definition of an even number ↗

⇒ A is not odd as we have demanded □

Mathematical induction

Def 46: To prove that $P(n)$ is true for all positive integers n , where $P(n)$ is a propositional function, we complete three steps:

- ① we verify $P(1)$ is true "basic step"
- ② we assume $P(k)$ "inductive step"
- ③ we show that the implication $P(k) \rightarrow P(k+1)$ is true for all positive integers k .

Example: Prove that for every positive integer n :

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

("Gaussian Sum formula")

$$P(1): 1 = \frac{1 \cdot (1+1)}{2} = \frac{1 \cdot 2}{2} = 1 \quad \checkmark$$

$$P(k) \quad 1+2+3+\dots+k = \frac{k(k+1)}{2}$$

inductive hypothesis must be used in the argument.

Let's take the next element to be added

$k+1$ must be added to the sequence

Use the IH (inductive hypothesis)

$$\begin{aligned} & \boxed{1+2+3+\dots+k} + (k+1) = \\ & = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \\ & = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2} \end{aligned}$$

in turn is: $\frac{n(n+1)}{2}$ when

$$\frac{\overset{n}{(k+1)} \overset{n}{(k+1+1)}}{2}$$

$$\boxed{2},$$

$$n = (k+1)$$

So, we showed:

$P(1)$ which is the "basic step"

$P(k) \rightarrow P(k+1)$ using $P(k)$ "inductive hypothesis"

and we correctly concluded $P(k+1)$.

So $\forall n (P(n))$. \square

Question: Can we use mathematical induction to prove properties about negative integers also (e.g. is there a situation where we have $P(k) \rightarrow P(k-1)$)?

Short answer: Yes, induction can be used for any set of objects which can be represented as a sequence.

E.g. you can use the induction twice:

- once to prove it for all positive integers
- once for all negative integers.

But, note that not all properties can be proven for negative integers.

Want more? see the link on math.stackexchange.com