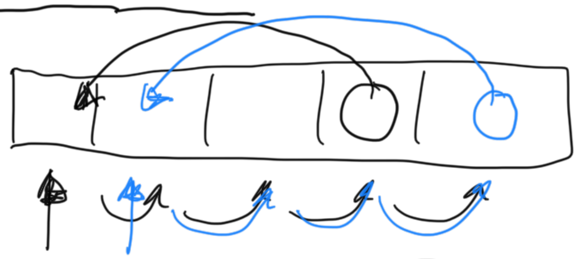
Question 5.3:

5.3 Look at the array below and consider how the array changes at each step of going through a selection-sort program, where the values of the array are being sorted in an increasing order (i.e. 1 2 3). Write down the state of the array BEFORE and AFTER each loop iteration of the program. What does the array look like after 3 iterations of selection-sort? (6pt)
Note that "after 3 iterations" means, when i=2, before entering the for loop and increasing i to 4. Assume that i begins with 0.

Array: 2 3 -5 4 3 -1 13

Selection sort



i=0: before: 2 3 -5 4 3 -1 13
1
after: -5 3 2 4 3 -1 13

i=1: before: -5 3 2 4 3 -1 13
2
after: -5 -1 2 4 3 3 13

i=2: before: -5 -1 2 4 3 3 13
3
after: -5 -1 2 4 3 3 13

-5 -1 2 4 3 3 13

i=3: before: -5 -1 2 4 3 3 13

after: −5 −1 2 3 4 3 13

i = 4: before: −5 −1 2 3 4 3 13

after: −5 −1 2 3 3 4 13

i = 5: before: −5 −1 2 3 3 4 13

after: −5 −1 2 3 3 4 13

```
/* a[0] to a[aLength-1] is the array to sort */
int i,j;
int aLength; // initialise to a's length

/* advance the position through the entire array */
/*   (could do i < aLength-1 because single element is also min element) */
for (i = 0; i < aLength-1; i++)
{
    /* find the min element in the unsorted a[i .. aLength-1] */

    /* assume the min is the first element */
    int jMin = i;
    /* test against elements after i to find the smallest */
    for (j = i+1; j < aLength; j++)
    {
        /* if this element is less, then it is the new minimum */
        if (a[j] < a[jMin])
        {
            /* found new minimum; remember its index */
            jMin = j;
        }
    }

    if (jMin != i)
    {
        swap(a[i], a[jMin]);
    }
}
```

$O(n)$

$O(n)$

$O(n^2)$

Worst case: $O(n^2)$

Best case: $O(n^2)$

$\Theta(n^2)$

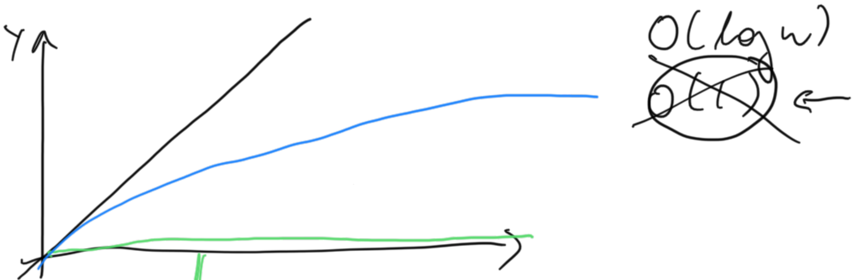
best case: ...

$O(\log n)$:

17 8 9 5 −1 14 ....

4

$O(n)$



4

−1 5 8 9 14 17 $O(n)$

3 2 1 6

−1 ≠ 4 3

$O(\log n)$ ← $O(n)$, $O(n^2)$

$O(1)$

K_0	V_0
K_1	V_1
K_2	V_2
K_3	V_3
K_4	V_4

id → student

17 → Alice

3 → Bob

4 → Fred

hash map(3) ⇒ "Bob"

−1	X
5	X
8	X
9	X
14	X
17	X

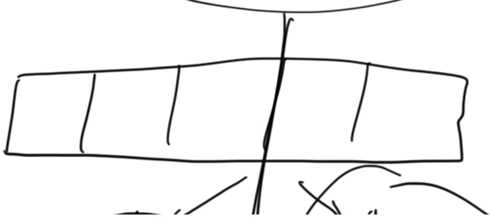
hashmap(4)

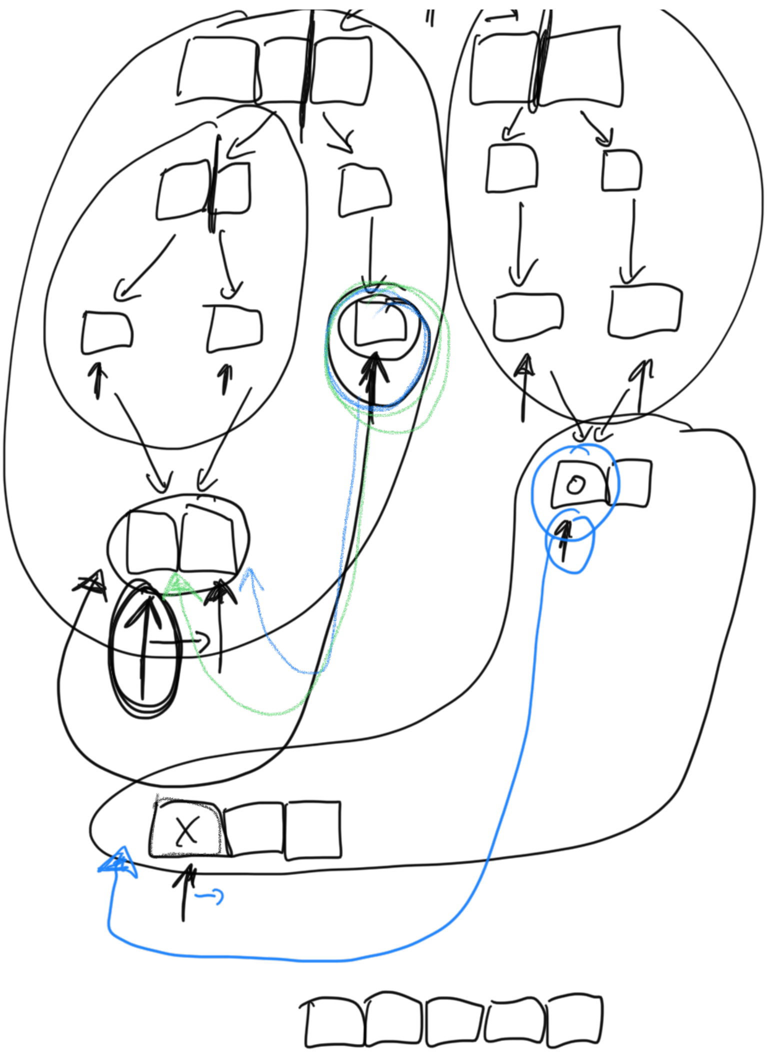
$O(1)$

$O(n^2)$ Selection sort

Merge sort $O(n \cdot \log n)$

$O(n)/O(n^2)$ Bubble sort





Question 4.3

4.3 Prove that each odd square results in the remainder 1 when divided by 8. (5pt)

odd number: $k = 2l + 1$ (definition)

$\Rightarrow$ square odd number: $k^2 = \underbrace{(2 \cdot l + 1)^2}$

$$= (2 \cdot l + 1) \cdot (2 \cdot l + 1)$$

$$= \underbrace{2 l \cdot 2 l} + \underbrace{2 l \cdot 1 + 1 \cdot 2 \cdot l} +$$

$$= 4 \cdot l^2 + 2 \cdot 2 \cdot l + 1$$

$= 4 \cdot \ell^2 + 4 \cdot \ell + 1$

$= 4 \cdot (\ell^2 + \ell) + 1$

$= 4\underbrace{\ell(\ell+1)} + 1$

case 1: ℓ is even: $\ell = 2p$

$\ell \cdot (\ell+1)$: $2p \cdot ((2p)+1)$

$= 4p^2 + 2p$

$= 2 \cdot 2p^2 + 2p$

$= \underbrace{2 \cdot (2p^2 + p)}$

$= 2 \cdot q \Rightarrow$ definition of an even number

case 2: ℓ is odd: $\ell = 2p+1$

$\ell \cdot (\ell+1)$: $(2p+1) \cdot ((2p+1)+1)$

$= (2p+1) \cdot (2p+2)$

$= 2 \cdot p \cdot 2 \cdot p + 2 \cdot 2p + 1 \cdot 2p + 1$

$= 4p^2 + 4p + 2p + 2$

$= 2 \cdot 2p^2 + 2 \cdot 2p + 2p + 2$

$= 2 \cdot \underbrace{(2p^2 + 2p + p + 1)}_{q}$

$= 2 \cdot q \Rightarrow$ even number.

from case 1 and case 2: $\ell \cdot (\ell+1)$ is always an even number

$\ell \cdot (\ell+1) = 2t$

$4\,\ell(\ell+1) + 1 \Rightarrow 4 \cdot 2t + 1$

$= 4 \cdot 2t \quad +1$

$= 8 \cdot t \quad +1$

□

Question 4.1

4.1 Prove that s = 1 + 2 + 3 + 4 + … + n = (1/2) * n * (n+1) (Gaussian formula). (4pt)

basic step : n=1

$$s = \underbrace{\boxed{1 + 2 + \ldots + n}}_{LHS} = \underbrace{\boxed{\frac{n \cdot (n+1)}{2}}}_{RHS}$$

LHS: 1

RHS: $\frac{n \cdot (n+1)}{2} = \frac{1 \cdot (1+1)}{2} = \frac{1 \cdot 2}{2} = \frac{2}{2} =$

LHS = RHS ✓

indnetive hypothesis

n = k

" formula holds for any given k "

(and we showed this that it holds for k = 1)

does it also hold for k's immediate succesor

$k \to k+1$?

from k=1 → k=2

from k=2 → k=3

from k=3 → k=4

" "

$x = \infty$

$S = 1 + 2 + 3 + \ldots + k = \frac{k \cdot (k+1)}{2}$

inductive step: $k \longrightarrow k+1$

LHS

$$\underbrace{(1+2+3+\ldots+k)}_{\text{using inductive hypothesis}} + (k+1)$$

$$\left(\frac{k \cdot (k+1)}{2}\right) + (k+1)$$

$$= \frac{k \cdot (k+1)}{2} + \frac{2 \cdot (k+1)}{2}$$

$$= \frac{k \cdot (k+1) + 2 \cdot (k+1)}{2}$$

$$= \frac{(k+1) \cdot (k+2)}{2}$$

RHS: $\frac{n \cdot (n+1)}{2}$ $\quad n: (k+1)$

$$\frac{(k+1) \cdot ((k+1)+1)}{2}$$

$$= \frac{(k+1) \cdot (k+2)}{2}$$

LHS = RHS $\square$

Question 4.2

4.2 Prove that for any real number $x > -1$ and any positive integer $n > 0$, $(1 + x)^n \geq 1 + nx$. (6pt)

Basic step: $n = 1$:

$$(1+x)^1 \geq 1 + 1 \cdot x$$
$$\Leftrightarrow \quad 1+x \geq 1+x \qquad \checkmark$$

<u>inductive hypothesis:</u>

$n = k$: formula holds for any given k.
(and we showed that it holds for $k=1$)

does it hold for k's immediate successor? $k \to k+1$?

if so, the formula must hold for all k's as we can construct:

from $k=1 \to k=2$
from $k=2 \to k=3$
from $k=3 \to k=4$
$\vdots$ "$k = \infty$"

$$\boxed{(1+x)^k \geq 1 + k \cdot x}$$ is assumed to h

<u>inductive step:</u> $k \to k+1$:

$$\text{"}\boxed{(1+x)^{(k+1)}} \geq \boxed{1 + (k+1) \cdot x}\text{" ?}$$

LHS RHS

$$\Downarrow$$

$$\boxed{(1+x)^{(k+1)}}$$
$$\Leftrightarrow \underbrace{\boxed{(1+x)^k}(1+x)}$$

using our inductive hypothesis, we know that $(1+x)^k$ is always larger or equal to $(1+k \cdot x)$; so we write:

LHS $$\boxed{(1+x)^k \cdot (1+x)} \geq \boxed{(1+k \cdot x)} \cdot (1+x)$$

rewrite

$$(1+k\cdot x)(1+x)$$
$$= \underbrace{1\cdot(1+x)} + (k\cdot x)\cdot(1+x)$$
$$= 1+x + k\cdot x + k\cdot x\cdot x$$
$$= 1+x + k\cdot x + k\cdot x^2$$
$$= \underbrace{(1+x + k\cdot x)}_{\text{"RHS"}} + k\cdot x^2$$

$k\cdot x^2$ we know that: $k>0$

(from the question: $n>0$)

$x>-1$: x^2 is 0 or great

$$(-0.99)^2 = (-1)^2\cdot(0.99)^2 > 0$$

$$\underbrace{k>0 \quad x^2 \geq 0}_{k\cdot x^2 \geq 0}$$

so far we know:

$$(1+x)^{k+1} = (1+x)^k(1+x) \geq \boxed{(1+x+k\cdot x)} + k\cdot x$$

≥ 0

how does the right hand side look like?

" remember: $1+n\cdot x$ "
$(k+1)$

$$1+(k+1)\cdot x$$
$$= \boxed{(1+k\cdot x+x)} = (1+x+k\cdot x)$$

so, combining LHS with RHS:

$$\boxed{(1+x)^{k+1}} = (1+x)^k(1+x) \geq (1+x+k\cdot x) + k\cdot x^2$$

LHS

is either equal or larger (due to $k \cdot x^2$) than

$\geq (1 + x + k \cdot x)$ RHS

$\square$

Question 2.5

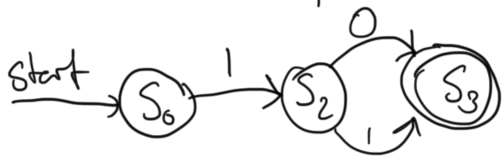
2.5 Draw a finite-state machine that recognizes the set: $(01 \cup 111)^*10^*(0 \cup 1)$ (13 pt)
Note that unreadable drawings will be awarded with 0 points.

$(01 \cup 111)^* 1 0^* (0 \cup 1)$

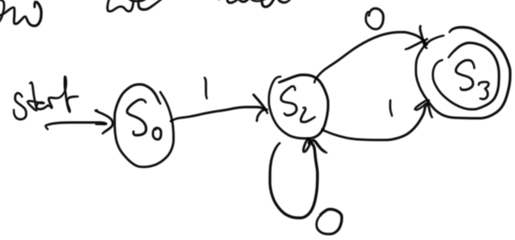
"or"

"zero times or more"

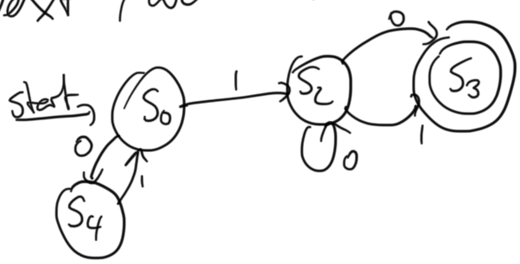
$\Rightarrow 1 \; (0 \cup 1)$



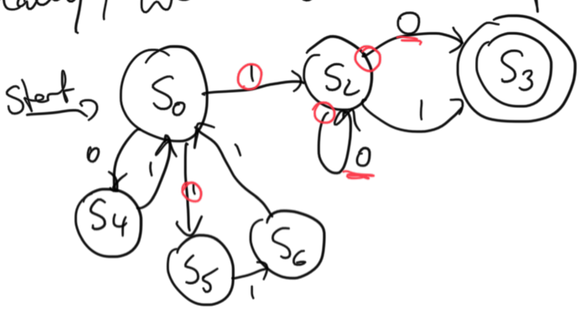
Now we add the center 0^*:



Next, we add the optional "01":



Finally, we add the optional "111":



$(01 \cup 111)^* 1 0^* (0 \cup 1$

Question 2.4

2.4 Add missing elements (states, transitions, or labels) or remove incorrect elements (states, transitions, or labels) in a deterministic finite-state automaton A4 so that it recognizes the set of all bit strings that begin and end with 11. (4pt)