Numerical methods and machine learning algorithms for solution of Inverse problems

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Microwave medical imaging in monitoring of hyperthermia: the course project and code

- Presentation of the course project *"Regularized algorithms for detection of tumours in microwave medical imaging"*. Matlab code (data and programs, zip file) with an example see in CANVAS, “Computer Projects”:

  `http://www.math.chalmers.se/Math/Grundutb/CTH/tma265/2021/IPcourse/Project_Hyperthermi.pdf`

- Matlab code: in CANVAS as well as in

  `https://github.com/ProjectWaves24/MicrowaveHyperMatlab`

- Advanced C++/PETSc computations and visualization in paraview: `https://github.com/ProjectWaves24/MESH`. Needs account in github, contact me.

- Advanced C++/PETSc computations using adaptive FEM:

  `https://github.com/ProjectWaves24/MicrowaveHypAFEM2`

  Needs account in github, contact me.

Statement of an ill-posed problem

Let $\Omega \subset \mathbb{R}^n, n = 2, 3$ which is a bounded domain with the boundary $\partial\Omega$. Our goal is to solve a Fredholm integral equation of the first kind

$$\int_\Omega \rho(x, y) z(x) dx = u(y) \quad y \in \Omega, \tag{1}$$

where $u(y) \in L_2(\Omega)$, $z(x) \in H$, $\rho(x, y) \in C^k(\Omega \times \Omega)\,, k \geq 0$ is the kernel of the integral equation. We can rewrite (1) in an operator form as

$$A(z) = u \tag{2}$$

with an operator $A : H \to L_2(\Omega)$ defined as

$$A(z) := \int_\Omega \rho(x, y) z(x) dx. \tag{3}$$

The Problem (P).
Let $z(x) \in H$ in

$$\int_\Omega \rho(x, y) z(x) dx = u(y) \quad y \in \Omega, \tag{4}$$

be unknown in Ω. Determine $z(x) \in H$ for $x \in \Omega$ assuming that functions $\rho(x, y) \in C^k(\Omega \times \Omega)\,, k \geq 0$ and $u(y) \in L_2(\Omega)$ in (4) are known.

The Tikhonov functional

Let W_1, W_2, Q be three Hilbert spaces, $Q \subseteq W_1$ as a set. We denote scalar products and norms in these spaces as

$$(\cdot,\cdot), \|\cdot\| \text{ for } W_1,$$

$$(\cdot,\cdot)_2, \|\cdot\|_2 \text{ for } W_2$$

$$\text{and } [\cdot,\cdot], [\cdot] \text{ for } Q.$$

Let $A : W_1 \to W_2$ be a bounded linear operator. Our goal is to find the function $z \in Q$ which minimizes the Tikhonov functional

$$J_\alpha(z) = \frac{1}{2}\|Az - u\|_2^2 + \frac{\alpha}{2}[z]^2, u \in W_2; z \in Q, \tag{5}$$

where $\alpha > 0$ is a regularization parameter. We search for a stationary point of the above functional with respect to z satisfying $\forall b \in Q$

$$J'_\alpha(z)(b) = 0, \tag{6}$$

where $J'_\alpha(z)$ is the Fréchet derivative of the functional (5).

The Tikhonov functional

When the operator $A : L_2 \to L_2$ the following Lemma is valid:
Lemma 1a [BKS] *Let* $A : L_2 \to L_2$ *be a bounded linear operator. Then the Fréchet derivative of the functional (5) is*

$$J'_\alpha(z)(b) = (A^*Az - A^*u, b) + \alpha[z, b], \forall b \in Q. \tag{7}$$

In particular, for the integral operator (4) we have

$$J'_\alpha(z)(b) = \int_\Omega b(s)\left[\int_\Omega z(y)\left(\int_\Omega \rho(x,y)\rho(x,s)\,dx\right)dy - \int_\Omega \rho(x,s)\,u(x)\,dx\right]ds \tag{8}$$
$$+\alpha[z, b], \forall b \in Q.$$

[BKS] A. B. Bakushinsky, M. Y. Kokurin, A. Smirnova, *Iterative methods for ill-posed problems*, Walter de Gruyter GmbH&Co., 2011.

The Tikhonov functional

When the operator $A : H^1 \to L_2$ the following Lemma is valid:
Lemma 1b [BGN] Let $A : H^1(\Omega) \to L_2(\Omega_\kappa)$ be a bounded linear operator. Then the Fréchet derivative of the functional

$$M_\alpha(f) = \frac{1}{2}\|Af - u\|^2_{L_2(\Omega_\kappa)} + \frac{\alpha}{2}\| \, |\nabla f| \, \|^2_{L^2(\Omega)}, \tag{9}$$

is

$$M'_\alpha(f)(b) = (A^*Af - A^*u, b) + \alpha(|\nabla f|, |\nabla b|), \ \forall b \in H^1(\Omega), \tag{10}$$

with a convex growth factor b, i.e., $|\nabla b| < b$

[BGN] L. Beilina, G. Guillot, K. Niinimäki„ The Finite Element Method and Balancing Principle for Magnetic Resonance Imaging, Springer Proceedings in Mathematics and Statistics, vol 328. Springer, Cham (2020).

Lemma 2 is also well known since $A : W_1 \to W_2$ is a bounded linear operator.

Lemma 2 [TGSY] *Let the operator $A : W_1 \to W_2$ be a bounded linear operator which has the Fréchet derivative of the functional (5). Then the functional $J_\alpha(z)$ is strongly convex on the space Q and*

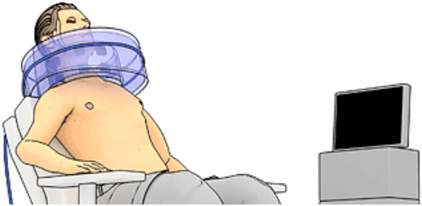
$$(J'_\alpha(x) - J'_\alpha(z), x - z) \geq \alpha[x - z]^2, \forall x, z \in Q.$$

It is known from the theory of convex optimization that Lemma 2 implies existence and uniqueness of the global minimizer $z_\alpha \in Q$ of the functional J_α such that

$$J_\alpha(z_\alpha) = \inf_{z \in Q} J_\alpha(z).$$

[TGSY] A.N. Tikhonov, A.V. Goncharsky, V.V. Stepanov and A.G. Yagola, *Numerical Methods for the Solution of Ill-Posed Problems*, London: Kluwer, London, 1995.

Microwave medical imaging in monitoring of hyperthermia



- Joint work with the group of Biomedical Imaging at the Department of Electrical Engineering at CTH, Chalmers.
- Microwave hyperthermia is used for cancer therapies: it increases the tumour temperature to $40 - 44^\circ C$ keeping healthy tissue at the normal temperature.
- Thermal dose monitoring is critical for treatment. Thus, robust real-time methods for localization of the focal point in the target are needed.
- AFEM with combination of least squares method is applied in microwave thermometry for non-invasive monitoring of hyperthermia [1].

[1] M. G. Aram, L. Beilina, H. Dobsicek Trefna, Microwave Thermometry with Potential Application in Non-invasive Monitoring of Hyperthermia, *Journal of Inverse and Ill-posed problems*, https://doi.org/10.1515/jiip-2020-0102, 2020.

CIP for electromagnetic problems. Maxwell's equations

Consider a region of space that has no electric or magnetic current sources, but may have materials that absorb electric or magnetic field energy. Then, using MKS units, the time-dependent Maxwell's equations are given in differential and integral form by *Faraday's law* :

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} - \mathbf{M} \tag{11a}$$

$$\frac{\partial}{\partial t} \iint_A \mathbf{B} \cdot \mathbf{dA} = -\oint_L \mathbf{E} \cdot \mathbf{dL} - \iint_A \mathbf{M} \cdot \mathbf{dA} \tag{11b}$$

The MKS system of units is a physical system of units that expresses any given measurement using fundamental units of the metre, kilogram, and/or second (MKS))

A. Taflove, S. C. Hagness, Computational Electromagnetics. The finite-difference time-domain method, 3rd edition, *Artech House Publishers*, 2005.

Maxwell's equations

Ampere's law :

$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J} \tag{12a}$$

$$\frac{\partial}{\partial t} \iint_A \mathbf{D} \cdot \mathbf{dA} = \oint_L \mathbf{H} \cdot \mathbf{dL} - \iint_A \mathbf{J} \cdot \mathbf{dA} \tag{12b}$$

Gauss' law for the electric field :

$$\nabla \cdot \mathbf{D} = 0 \tag{13a}$$

$$\oiint_A \mathbf{D} \cdot \mathbf{dA} = 0 \tag{13b}$$

Gauss' law for the magnetic field :

$$\nabla \cdot \mathbf{B} = 0 \tag{14a}$$

$$\oiint_A \mathbf{B} \cdot \mathbf{dA} = 0 \tag{14b}$$

In (11) to (14), the following symbols (and their MKS units) are defined:

E	:	electric field (volts/meter)
D	:	electric flux density (coulombs/meter^2)
H	:	magnetic field (amperes/meter)
B	:	magnetic flux density (webers/meter^2)
A	:	arbitrary three-dimensional surface
dA	:	differential normal vector that characterizes surface A (meter^2)
L	:	closed contour that bounds surface A (volts/meter)
dL	:	differential length vector that characterizes contour L (meters)
J	:	electric current density (amperes/meter^2)
M	:	equivalent magnetic current density (volts/meter^2)

Maxwell's equations

In linear, isotropic, nondispersive materials (i.e. materials having field-independent, direction-independent, and frequency-independent electric and magnetic properties), we can relate **D** to **E** and **B** to **H** using simple proportions:

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_r \varepsilon_0 \mathbf{E}; \quad \mathbf{B} = \mu \mathbf{H} = \mu_r \mu_0 \mathbf{H} \tag{15}$$

where

ε	:	electrical permittivity (farads/meter)
ε_r	:	relative permittivity (dimensionless scalar)
ε_0	:	free-space permittivity (8.854×10^{-12} farads/meter)
μ	:	magnetic permeability (henrys/meter)
μ_r	:	relative permeability (dimensionless scalar)
μ_0	:	free-space permeability ($4\pi \times 10^{-7}$ henrys/meter)

Note that **J** and **M** can act as *independent sources* of E- and H-field energy, $\mathbf{J}_{source}$ and $\mathbf{M}_{source}$.

Maxwell's equations

We also allow for materials with isotropic, nondispersive electric and magnetic losses that attenuate E- and H-fields via conversion to heat energy. This yields

$$\mathbf{J} = \mathbf{J}_{source} + \sigma \mathbf{E}; \quad \mathbf{M} = \mathbf{M}_{source} + \sigma^* \mathbf{H} \tag{16}$$

where σ : electric conductivity (siemens/meter)
σ^* : equivalent magnetic loss (ohms/meter)

Finally, we substitute (15) and (16) into (11a) and (12a). This yields Maxwell's curl equations in linear, isotropic, nondispersive, lossy materials:

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \mathbf{E} - \frac{1}{\mu} \left(\mathbf{M}_{source} + \sigma^* \mathbf{H} \right) \tag{17}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon} \nabla \times \mathbf{H} - \frac{1}{\varepsilon} \left(\mathbf{J}_{source} + \sigma \mathbf{E} \right) \tag{18}$$

Write now Maxwell's curl equations in linear, isotropic, nondispersive, lossy materials with $\sigma^* = 0, \mathbf{M}_{source} = 0$:

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \mathbf{E} \tag{19}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon} \nabla \times \mathbf{H} - \frac{1}{\varepsilon} \sigma \mathbf{E} - \frac{1}{\varepsilon} \mathbf{J}_{source} \tag{20}$$

Taking now $\frac{\partial}{\partial t}$ from (20) and multiplying by ε, and then taking $\nabla \times$ from (19), we have:

$$\nabla \times \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \frac{1}{\mu} \nabla \times \mathbf{E} \tag{21}$$

$$\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\partial}{\partial t} \nabla \times \mathbf{H} - \sigma \frac{\partial}{\partial t} \mathbf{E} - \frac{\partial}{\partial t} \mathbf{J}_{source} \tag{22}$$

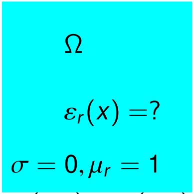
CIPs for electric wave propagation

Substitude the right hand side of (21) into (22) instead of $\frac{\partial}{\partial t} \nabla \times \mathbf{H}$ to obtain Maxwell's equations for electric field $\mathbf{E} = (E_1, E_2, E_3)$. Let us consider now Cauchy problem for the Maxwell's equations for electric field $\mathbf{E}$ in the domain $\Omega_T = \Omega \times [0, T]$:

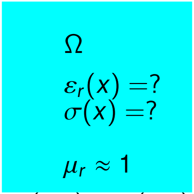
$$
\begin{aligned}
\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \nabla \times \frac{1}{\mu} \nabla \times \mathbf{E} = -\sigma \frac{\partial}{\partial t} \mathbf{E} - \frac{\partial}{\partial t} \mathbf{J}_{source} \text{ in } \ \Omega_T, \\
\nabla \cdot (\varepsilon \mathbf{E}) = 0, \\
\mathbf{E}(\mathbf{x}, \mathbf{0}) = f_0(x), \quad \mathbf{E_t}(\mathbf{x}, \mathbf{0}) = f_1(x) \text{ in } \ \Omega,
\end{aligned}
\tag{23}
$$

- Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$ and specify time variable $t \in [0, T]$. Next, we supply the Cauchy problem by the appropriate b.c.
- $\varepsilon(x)$ and $\sigma(x)$ are dielectric permittivity and electric conductivity functions, respectively of the domain Ω. In (23), $\varepsilon(x) = \varepsilon_r(x)\varepsilon_0, \mu = \mu_r \mu_0$ and $\sigma(x)$ are dielectric permittivity, permeability and electric conductivity functions, respectively, ε_0, μ_0 are dielectric permittivity and permeability of free space, respectively.

CIPs for electric wave propagation



$E(x,t) = g(x,t)$ on $\partial\Omega$



$E(x,t) = g(x,t)$ on $\partial\Omega$

Inverse Problem (EIP1) *Determine the relative dielectric permittivity function $\varepsilon_r(x)$ in Ω for $x \in \Omega$ in nonconductive ($\sigma(x) = 0$) and nonmagnetic ($\mu_r = 1$) media when the measured function $g(x,t)$ s.t.*

$$\mathbf{E}(x,t) = g(x,t), \forall (x,t) \in \partial\Omega \times (0,T].$$

is known in Ω.

Inverse Problem (EIP2) *Determine the functions $\epsilon(x), \sigma(x)$ in Ω for $x \in \Omega$ for $\mu_r \approx 1$ in water assuming that $g(x,t)$ is known in $\partial\Omega \times (0,T]$.*

Maxwell's equations in frequency domain

Assuming $\mathbf{E}(\mathbf{x}, \mathbf{t}) = \widehat{E}(x, \omega) \cdot e^{-i\omega t}$ and $\mathbf{J}_{source} = \widehat{J}(x, \omega) \cdot e^{-i\omega t}$ and applying this to (23) with $\mu_r = 1$ we obtain the following vector wave equation:

$$\nabla \times \nabla \times \widehat{E}(x, \omega) - \omega^2 \left(\frac{\varepsilon_r(x)}{c^2} + i\mu_0 \, \frac{\sigma(x)}{\omega} \right) \widehat{E}(x, \omega) = i\omega\mu_0 \widehat{J}(x, \omega). \quad (24)$$

We introduce the spatially distributed complex dielectric function $\varepsilon'(x)$:

$$\varepsilon'(x) = \varepsilon_r(x) \frac{1}{c^2} + i\mu_0 \frac{\sigma(x)}{\omega}, \quad (25)$$

where ω is the angular frequency. Then the equation (31) transforms to the equation

$$\nabla \times \nabla \times \widehat{E}(x, \omega) - \omega^2 \varepsilon'(x) \widehat{E}(x, \omega) = i\omega\mu_0 \widehat{J}(x, \omega). \quad (26)$$

which should be supplied by appropriate boundary conditions.

Applying $\nabla \times \nabla \times \widehat{E} = \nabla(\nabla \cdot \widehat{E}) - \nabla \cdot (\nabla \widehat{E})$ and in case of $\mathbf{E}(\mathbf{x}, \mathbf{t}) = \widehat{E}(x, \omega) \cdot e^{i\omega t}$ we obtain inhomogeneous Helmholtz equation

$$\triangle \widehat{E} + k^2 \widehat{E} = i\omega\mu_0 \widehat{J}, \tag{27}$$

where $k^2 = \omega^2 \varepsilon'$. This equation can be rewritten for the solution $\widehat{E} = E(r)$ in cylindrical coordinates and in transverse electric (TE) mode as a Bessel equation

$$\left(\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial}{\partial r}) + k^2 \right) E = i\omega\mu_0 J. \tag{28}$$

The general solution to this equation is in the form

$$E(r) = AJ_0(kr) + BN_0(kr), \tag{29}$$

where J_o and N_0 are zero-order Bessel's functions of the first and second order, respectively. The time-harmonic solution of the equation (28) is given by

$$E(r, \omega) := E(r) = -\frac{\omega\mu_r}{4} \int_S J H_0^{(2)}(kR) \, dS, \tag{30}$$

for a generalized source initialized at r_0 and $R = |r - r_0| = \sqrt{r^2 + r_0{}^2 - 2rr_0 cos(\varphi - \varphi_0)}$.

[BE] L. Beilina and A. Eriksson, Reconstruction of dielectric constants in a cylindrical waveguide, *Inverse Problems and Applications, Springer Proceedings in Mathematics & Statistics*, Vol. 120, 2015.

Derivation of the volume integral equation

Let $r = (x, y, z)$.
Model PDE in a non-magnetic medium with the Silver-Müller radiation condition at infinity:

$$\nabla \times \nabla \times \widehat{E}(r) - \omega^2 \left(\frac{\varepsilon_r(r)}{c^2} + i\mu_0 \frac{\sigma(r)}{\omega} \right) \widehat{E}(r) = i\omega\mu_0 \widehat{J}(r), \quad r \in \mathbb{R}^3. \tag{31}$$

which we rewrite as

$$\nabla \times \nabla \times \widehat{E}(r) - \omega^2 \varepsilon'(r) \widehat{E}(r) = i\omega\mu_0 \widehat{J}(r). \tag{32}$$

[BAK] L. Beilina, M. G. Aram, E. Karchevskii, An adaptive finite element method for solving 3D electromagnetic volume integral equation with applications in microwave thermometry, Journal of Computational Physics, 2022.

Derivation of the volume integral equation

Subtracting the term $\nabla \times \nabla \times \widehat{E}(r) - \omega^2 \varepsilon_b \widehat{E}(r)$ from both sides of (32), we get

$$-\omega^2(\varepsilon'(r) - \varepsilon_b)\widehat{E}(r) = i\omega\mu_0 \widehat{J}(r) - \nabla \times \nabla \times \widehat{E}(r) + \omega^2 \varepsilon_b \widehat{E}, \tag{33}$$

which can be rewritten as

$$\nabla \times \nabla \times \widehat{E}(r) - \omega^2(\varepsilon'(r) - \varepsilon_b)\widehat{E}(r) = i\omega\mu_0 \widehat{J}(r) + \omega^2 \varepsilon_b \widehat{E}. \tag{34}$$

Here, ε_b is the dielectric permittivity corresponding to the background medium.

Derivation of the volume integral equation

We introduce the dyadic Green's function $\bar{G}(r, r')$ for the problem in a homogeneous medium (see, for example, [C]):

$$\nabla \times \nabla \times \bar{G}(r, r') - \omega^2 \varepsilon_b \bar{G}(r, r') = I\delta(r - r'), \quad r \in \mathbb{R}^3, \tag{35}$$

where I is an identity operator. Rewriting equation (34) according to the form of equation (35), we have

$$\nabla \times \nabla \times \widehat{E}(r) - \omega^2 \varepsilon_b \widehat{E}(r) = i\omega\mu_0 \widehat{J}(r) + \omega^2 (\varepsilon'(r) - \varepsilon_b)\widehat{E}(r). \tag{36}$$

We multiply (36) by $\bar{G}(r, r')$ and (35) by $\widehat{E}(r)$, correspondingly, to get:

$$\begin{aligned} &\nabla \times \nabla \times \widehat{E}(r)\bar{G}(r, r') - \omega^2 \varepsilon_b \widehat{E}(r)\bar{G}(r, r') \\ &= i\omega\mu_0 \widehat{J}(r)\bar{G}(r, r') + \omega^2 (\varepsilon'(r) - \varepsilon_b)\widehat{E}(r)\bar{G}(r, r'), \end{aligned} \tag{37}$$

$$\nabla \times \nabla \times \bar{G}(r, r')\widehat{E}(r) - \omega^2 \varepsilon_b \bar{G}(r, r')\widehat{E}(r) = \delta(r - r')\widehat{E}(r). \tag{38}$$

[C] W. C. Chew, *Waves and fields in inhomogeneous media*, New York, IEEE Press, 1995.

Then subtracting (38) from (37) and integrating, we obtain

$$
\begin{aligned}
&(\nabla \times \nabla \times \widehat{E}(r), \bar{G}(r,r')) - (\omega^2 \varepsilon_b \widehat{E}(r), \bar{G}(r,r')) - (\nabla \times \nabla \times \bar{G}(r,r'), \widehat{E}(r)) \\
&+ (\omega^2 \varepsilon_b \bar{G}(r,r'), \widehat{E}(r)) \\
&= (i\omega\mu_0 \widehat{J}(r), \bar{G}(r,r')) + (\omega^2(\varepsilon'(r) - \varepsilon_b)\widehat{E}, \bar{G}(r,r')) - (\delta(r-r'), \widehat{E}(r)).
\end{aligned}
\tag{39}
$$

Here, $(\cdot,\cdot)$ denotes the standard scalar product in space.
The above equation is reduced to the following equation

$$
\begin{aligned}
&(\nabla \times \nabla \times \widehat{E}(r), \bar{G}(r,r')) - (\nabla \times \nabla \times \bar{G}(r,r'), \widehat{E}(r)) \\
&= (i\omega\mu_0 \widehat{J}(r), \bar{G}(r,r')) + (\omega^2(\varepsilon'(r) - \varepsilon_b)\widehat{E}, \bar{G}(r,r')) - (\delta(r-r'), \widehat{E}(r)).
\end{aligned}
\tag{40}
$$

Derivation of the volume integral equation

We use integration by parts for the two terms on the left hand side of (40), and apply the Silver-Müller radiation condition at infinity, to get:

$$\begin{aligned}(\nabla \times \nabla \times \widehat{E}(r), \bar{G}(r, r')) &= (\nabla \times \widehat{E}(r), \nabla \times \bar{G}(r, r')),\\ (\nabla \times \nabla \times \bar{G}(r, r'), \widehat{E}(r)) &= (\nabla \times \bar{G}(r, r'), \nabla \times \widehat{E}(r)).\end{aligned} \tag{41}$$

Applying (41) in the left hand side of (40), we obtain

$$(\nabla \times \nabla \times \widehat{E}(r), \bar{G}(r, r')) - (\nabla \times \nabla \times \bar{G}(r, r'), \widehat{E}(r)) = 0. \tag{42}$$

Derivation of the volume integral equation

Using the principle of linear superposition, see details in [C], we finally get from (40)

$$\widehat{E}(r) = i\omega\mu_0 \int_\Omega \widehat{J}(r)\bar{G}(r,r')\, dr + \omega^2 \int_\Omega (\varepsilon'(r) - \varepsilon_b)\widehat{E}(r)\bar{G}(r,r')\, dr. \quad (43)$$

The first term in the right hand side of the above equation corresponds to the incident electric field $\widehat{E}_{inc}$, see [C]. Hence, (43) becomes

$$\widehat{E}(r) = \widehat{E}_{inc} + \omega^2 \int_\Omega (\varepsilon'(r) - \varepsilon_b)\widehat{E}(r)\bar{G}(r,r')\, dr. \quad (44)$$

[C] W. C. Chew, *Waves and fields in inhomogeneous media*, New York, IEEE Press, 1995.

Derivation of the volume integral equation

According to [HM], the scattered field is defined as $\widehat{E}_{sca} = \widehat{E}(r) - \widehat{E}_{inc}$ such that using the reciprocity $\bar{G}(r', r) = \bar{G}(r, r')$ and $O(r) = \varepsilon'(r) - \varepsilon_b(r)$, equation (44) transforms to

$$\widehat{E}_{sca} = \omega^2 \int_{\Omega} \widehat{E}(r) O(r) \bar{G}(r', r) \, dr. \tag{45}$$

which is the model volume integral equation.

[HM] M. Haynes, M. Moghaddam, Vector Green's function fopr S-parameter measurements of the electromagnetic volume integral equation, *IEEE Transactions on Antennas and Propagation*,vol. 60, No. 3, pp. 1400-1413, DOI: 0.1109/tap.2011.2180324, 2012.

Let we have a bi-static pair $(i,\ j)$ of antennas located on the scan line Γ, i.e. $\mathbf{r_i},\ \mathbf{r_j}\ \in\ \Gamma$.

Using Lorentz reciprocity theorem and under Born approximation, the scattered electric field between the pair of antennas at angular frequency of ω can be written as

$$\mathbf{E}_{ji}^{s} \simeq i\omega\mu_0 k_b^2 I(\omega) \int_{\Omega} \overline{\mathbf{G}}(\mathbf{r_j}, \mathbf{r}', \omega) \cdot \epsilon'(\mathbf{r}', \omega) \overline{\mathbf{G}}(\mathbf{r_i}, \mathbf{r}', \omega) dv' \tag{46}$$

where Ω is the imaging domain, $I(\omega)$ is the excitation current of the transmitter, k_b is the lossless background wavenumber, $\overline{\mathbf{G}}$ is the dyadic Green's function and ε' is defined as in (25).

Next, scattered fields $\mathbf{E}_{ji}^{s}$ are replaced with their corresponding S-parameters, as well as input power and characteristic impedance of the ports. Then (46) is transformed to the equation

$$S_{ji}^{sca}(\omega) \simeq C \int_{\Omega} \mathbf{E}_{inc,j}^{CST}(\mathbf{r}', \omega) \cdot \Delta O(\mathbf{r}', \omega) \mathbf{E}_{inc,i}^{CST}(\mathbf{r}', \omega) dv' \tag{47}$$

where $C = -k_b^2/(4i\omega\mu)$ and $\mathbf{E}_{inc,i}^{CST}$ is the exported E-field from CST under irradiation of the i^{th} antenna. Here, $\Delta O = \varepsilon'(\mathbf{r}) - \varepsilon_b'(\mathbf{r})$, $\varepsilon_b'(\mathbf{r})$ is baseline.

Equation (47) is the standard Fredholm integral equation of the first kind, and thus, it is an ill-posed problem. It can be solved for an linear operator A by minimizing the Tikhonov regularization functional

$$F(\varepsilon') = \frac{1}{2}\left\|A\varepsilon' - d\right\|^2_{L_2(\Omega)} + \frac{\lambda}{2}\left\|\varepsilon'\right\|^2_{L_2(\Omega)}. \tag{48}$$

where $d = S^{sca}$, λ is the regularization parameter. The optimal value will be:

$$F'(\varepsilon') = A^*A\varepsilon' - A^*d + \lambda\varepsilon' = 0. \tag{49}$$

Discretizing operator A, we get the matrix $\mathbf{A}$ and the problem (49) will be rewritten as the system of normal equations

$$\varepsilon' = (\mathbf{A}^T\mathbf{A} + \lambda I)^{-1}\mathbf{A}^T d. \tag{50}$$

Applying SVD of $\mathbf{A} = U\Sigma V^T$ in we get the equation to reconstruct ε' :

$$\varepsilon' = V(\Sigma^2 + \lambda I)^{-1}\Sigma U^T d. \tag{51}$$

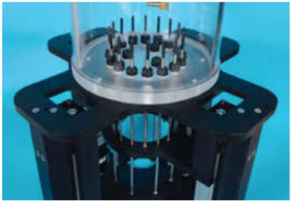
Applying SVD of $\mathbf{A} = U\Sigma V^T$ in we get the equation to reconstruct ε' :

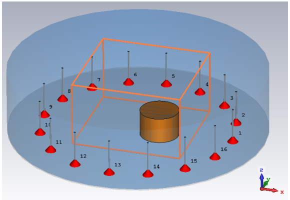
$$\varepsilon' = V(\Sigma^2 + \lambda I)^{-1}\Sigma U^T d. \tag{52}$$

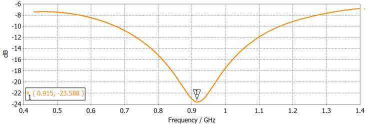
Proof: Since $\mathbf{A} = U\Sigma V^T$ then $\mathbf{A}^T = (U\Sigma V^T)^T = V\Sigma U^T$, then equation (28) can be written as:

$$\varepsilon' = (\mathbf{A}^T\mathbf{A}+\lambda I)^{-1}\mathbf{A}^T d = (V\Sigma U^T U\Sigma V^T+\lambda I)^{-1} V\Sigma U^T d = V(\Sigma^2+\lambda I)^{-1}\Sigma U^T d. \tag{53}$$

Reconstruction of heated target







Timeline (min)	$\varepsilon_r(t)$	$\sigma(t)$ (S/m)
t=0 (Baseline)	26	0.12
t=2	28	0.15
t=4	29	0.19
t=6	29.8	0.21
t=8	30.5	0.23
t=10	31	0.24
Coupling fluid 20:7 isopropyl : water ($\varepsilon_r = 24.5, \sigma = 0.46$)		

Microwave imaging for breast cancer detection. Top left: setup of the representation and actual photograph of the data acquisition platform for breast cancer detection used at CTH and Medfield Diagnostics AB: Assembled antenna hardware. Top right: schematic 3-D representation of 16 monopole antennas in a matching liquid tank, in CST(`http://www.cst.com`*); Bottom left: Return loss S_{11} of the designed antenna for the frequency band* 915 *MHz. Bottom right: permittivity and conductivity of the target as it starts to cool down from 55° C to 29° C over a ten-minute window of time.*

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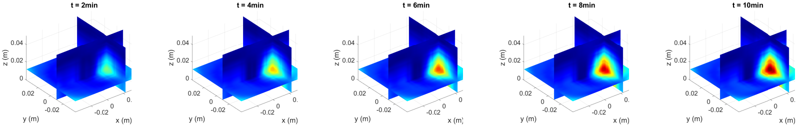
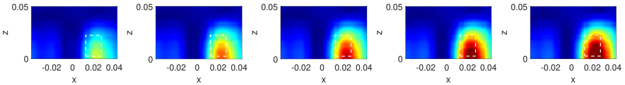
Reconstruction of heated target: least squares solution

Geometry with $nno = 40 \times 42 \times 26 = 43680$.
Solution is obtained via the formula

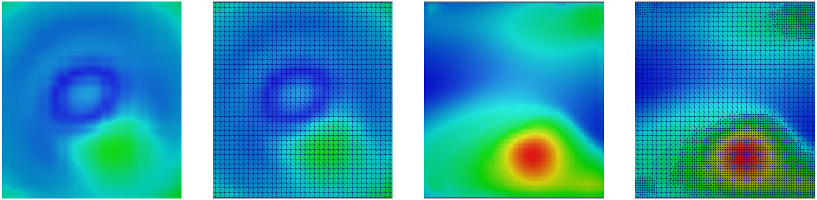
$$\varepsilon' = (\mathbf{A}^T\mathbf{A}+\lambda I)^{-1}\mathbf{A}^T d = (V\Sigma U^T U\Sigma V^T+\lambda I)^{-1} V\Sigma U^T d = V(\Sigma^2+\lambda I)^{-1}\Sigma U^T d$$

with $\lambda = 1$.

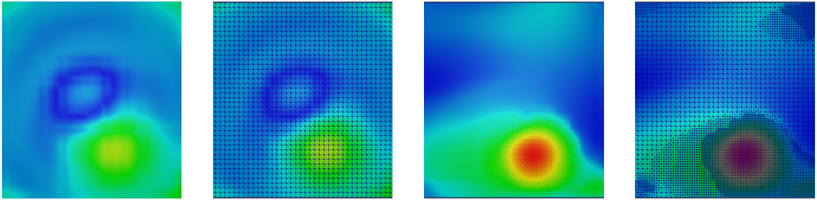




Reconstruction: Least Squares + AFEM, *xy*-plane

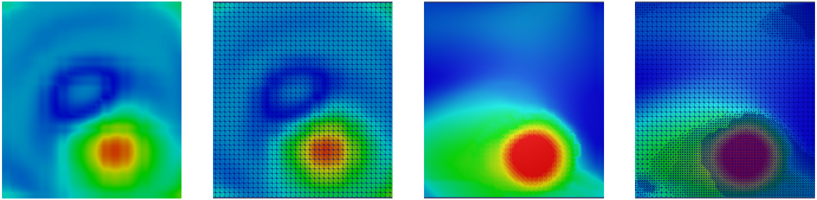


t = 2 min

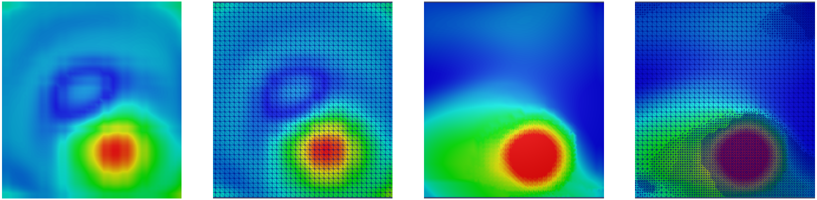


t = 4 min

Reconstruction: Least Squares + AFEM, *xy* plane

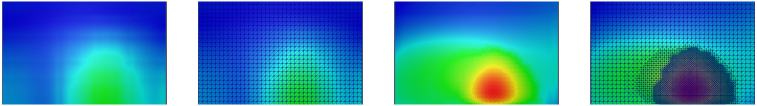


t = 8 min

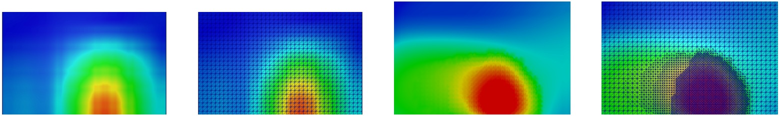


t = 10 min

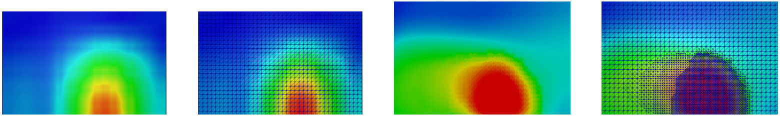
Reconstruction: Least Squares + AFEM, *zx* plane



t = 2 min



t = 8 min



t = 10 min

Convergence of fixed point algorithm and AFEM

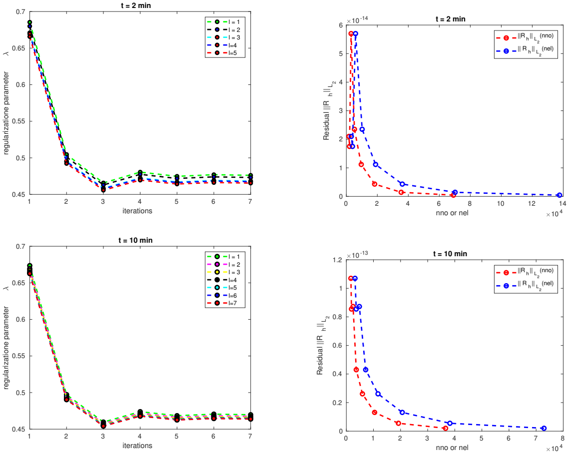


Figure: Left figures: convergence of fixed point algorithm. Here, *l* is the number of mesh refinement. Right figures: convergence of AFEM on adaptive locally refined meshes.

Project: Regularized algorithms for detection of tumours in microwave medical imaging

- In this project we will study different regularization strategies for detection of tumours using microwaves. This problem is a typical Coefficient Inverse Problem (CIP) for determination of complex dielectric permittivity function in Helmholtz equation from scattered electric field in frequency domain.
- Alternatively, the dielectric permittivity function can be determined from the solution of a Fredholm integral equation of the first kind which is an ill-posed problem.
- The goal of the current project is further development of mathematical methods presented in the recent paper [ABD] for real-life applications in microwave medical imaging.

M. G. Aram, L. Beilina, H. Dobsicek Trefna, Microwave Thermometry with Potential Application in Non-invasive Monitoring of Hyperthermia, *Journal of Inverse and Ill-posed problems*, 2020. https://doi.org/10.1515/jiip-2020-0102

Project: Regularized algorithms for detection of tumours in microwave medical imaging

More precisely, in this project students will:

- Study different regularized formulations of the reconstruction problem presented in the paper [ABD] which can be downloaded from the link

  `https://doi.org/10.1515/jiip-2020-0102`

- Determine the dielectric permittivity function by solving the regularized linear system of equations (LSE) in 3D by modifying existing Matlab code used for computations in the paper [ABD] available for download at

  `http://www.math.chalmers.se/Math/Grundutb/CTH/tma265/2021/IPcourse/MatlabCode_MicrowaveImaging.zip`.

  `https://github.com/ProjectWaves24/MicrowaveHyperMatlab`

Project: Regularized algorithms for detection of tumours in microwave medical imaging

- Test different regularization strategies (Morozov's discrepancy principle, Balancing principle) for choosing the regularization parameter λ.
- Test reconstructions with choosing different regularization terms, i.e. try to minimize

$$F(\varepsilon') = \frac{1}{2} \left\| A\varepsilon' - d \right\|^2_{L_2(\Omega)} + \frac{\lambda}{2} \left\| \nabla \varepsilon' \right\|^2_{L_2(\Omega)}. \tag{54}$$

or minimize

$$F(\varepsilon') = \frac{1}{2} \left\| A\varepsilon' - d \right\|^2_{L_2(\Omega)} + \frac{\lambda}{2} \left\| \varepsilon' + \nabla \varepsilon' \right\|^2_{L_2(\Omega)}. \tag{55}$$