

# Numerical methods and machine learning algorithms for solution of Inverse problems

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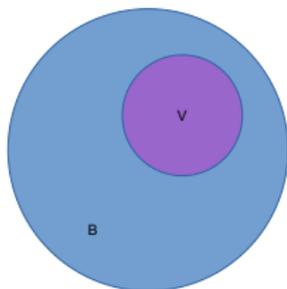
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# Physical formulations leading to ill- and well-posed problems

## Lecture 2

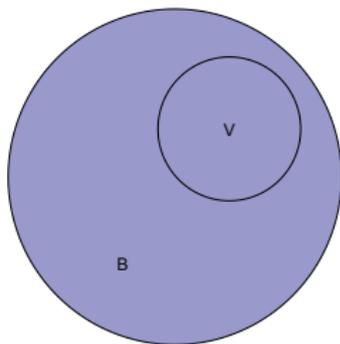
# Notations and Definitions



**Definition 1.** Let  $B$  be a Banach space. The set  $V \subset B$  is called *precompact* set if every sequence  $\{x_n\}_{n=1}^{\infty} \subseteq V$  contains a fundamental subsequence (i.e., the Cauchy subsequence).

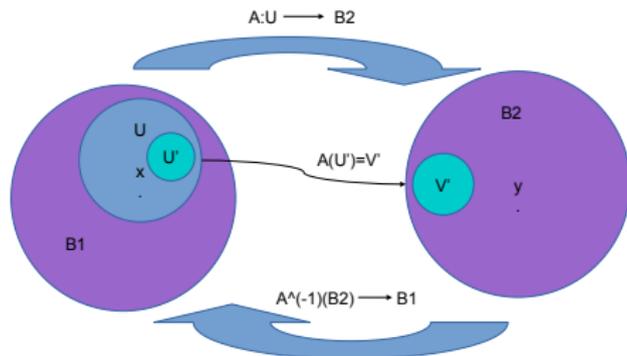
Although by the Cauchy criterion the subsequence in this Definition 1 converges to a certain point, there is no guarantee that this point belongs to the set  $V$ . If we consider the closure of  $V$ , i.e. the set  $\overline{V}$ , then all limiting points of all convergent sequences in  $V$  would belong to  $\overline{V}$ .

## Notations and Definitions



**Definition 2.** Let  $B$  be a Banach space. The set  $V \subset B$  is called *compact set* if  $V$  is a closed set,  $V = \overline{V}$ , every sequence  $\{x_n\}_{n=1}^{\infty} \subseteq V$  contains a fundamental subsequence and the limiting point of this subsequence belongs to the set  $V$ .

## Notations and Definitions



**Definition 3.** Let  $B_1$  and  $B_2$  be two Banach spaces,  $U \subseteq B_1$  be a set and  $A : U \rightarrow B_2$  be a continuous operator. The operator  $A$  is called a *compact operator* or *completely continuous* operator if it maps any bounded subset  $U' \subseteq U$  in a precompact set in  $B_2$ . Clearly if  $U'$  is a closed set, then  $A(U')$  is a compact set.

## Notations and Definitions. Ascoli-Archela theorem

The following theorem is well known under the name of **Ascoli-Archela theorem** (More general formulations of this theorem can also be found).

**Theorem** *The set of functions  $\mathcal{M} \subset C(\overline{\Omega})$  is a compact set if and only if it is uniformly bounded and equicontinuous. In other words, if the following two conditions are satisfied:*

1. *There exists a constant  $M > 0$  such that*

$$\|f\|_{C(\overline{\Omega})} \leq M, \quad \forall f \in \mathcal{M}.$$

2. *For any  $\varepsilon > 0$  there exists  $\delta = \delta(\varepsilon) > 0$  such that*

$$|f(x) - f(y)| < \varepsilon, \quad \forall x, y \in \{|x - y| < \delta\} \cap \overline{\Omega}, \quad \forall f \in \mathcal{M}.$$

# Classical Correctness and Conditional Correctness

The notion of the classical correctness is called sometimes *Correctness by Hadamard*.

**Definition.** Let  $B_1$  and  $B_2$  be two Banach spaces. Let  $G \subseteq B_1$  be an open set and  $F : G \rightarrow B_2$  be an operator. Consider the equation

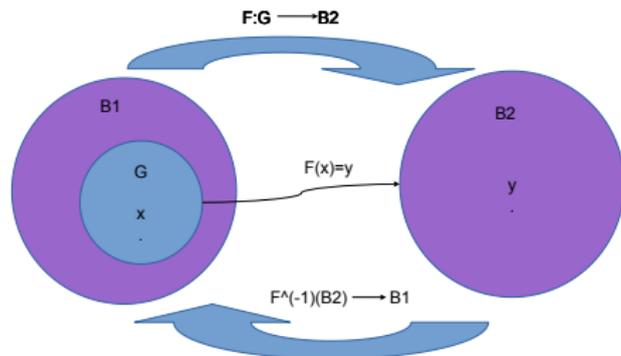
$$F(x) = y, \quad x \in G. \quad (1)$$

The problem of solution of equation (1) is called *well-posed by Hadamard*, or simply *well-posed*, or *classically well-posed* if the following three conditions are satisfied:

1. For any  $y \in B_2$  there exists a solution  $x = x(y)$  of equation (1) (existence theorem).
2. This solution is unique (uniqueness theorem).
3. The solution  $x(y)$  depends continuously on  $y$ . In other words, the operator  $F^{-1} : B_2 \rightarrow B_1$  is continuous.

If equation (1) does not satisfy to at least one these three conditions, then the problem (1) is called *ill-posed*.

## Classical Correctness



The problem is *classically well-posed* if:

1. For any  $y \in B_2$  there exists a solution  $x = x(y)$  of  $F(x) = y$ .
2. This solution is unique (uniqueness theorem).
3. The solution  $x(y)$  depends continuously on  $y$ . In other words, the operator  $F^{-1} : B_2 \rightarrow B_1$  is continuous.

# Classical Correctness and Conditional Correctness

We say that the right hand side of equation

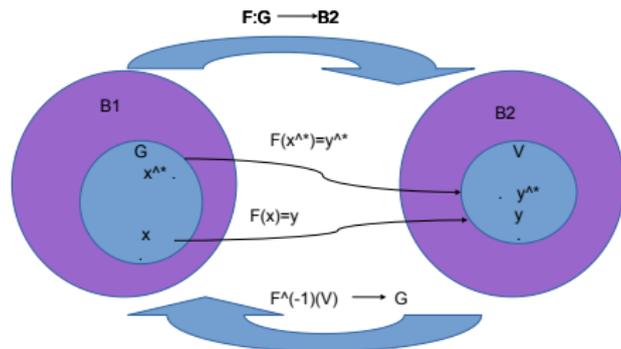
$$F(x) = y, \quad x \in G. \quad (2)$$

is given with an error of the level  $\delta > 0$  (small) if  $\|y^* - y\|_{B_2} \leq \delta$ , where  $y^*$  is the exact value.

**Definition** Let  $B_1$  and  $B_2$  be two Banach spaces. Let  $G \subset B_1$  be an *a priori* chosen set of the form  $G = \overline{G_1}$ , where  $G_1$  is an open set in  $B_1$ . Let  $F : G \rightarrow B_2$  be a continuous operator. Suppose that  $\|y^* - y_\delta\|_{B_2} \leq \delta$ . Here  $y^*$  is the ideal noiseless data,  $y_\delta$  is noisy data. The problem (2) is called *conditionally well-posed on the set*  $G$ , or *well-posed by Tikhonov* on the set  $G$  if the following three conditions are satisfied:

1. It is *a priori* known that there exists an ideal solution  $x^* = x^*(y^*) \in G$  of this problem for the ideal noiseless data  $y^*$ .
2. The operator  $F : G \rightarrow B_2$  is one-to-one.
3. The inverse operator  $F^{-1}$  is continuous on the set  $F(G)$ .

# Conditional Correctness



The problem (2) is called *conditionally well-posed on the set*  $G$  if:

1. It is *a priori* known that there exists an ideal solution  $x^* = x^*(y^*) \in G$  of this problem for the ideal noiseless data  $y^*$ .
2. The operator  $F : G \rightarrow B_2$  is one-to-one.
3. The inverse operator  $F^{-1}$  is continuous on the set  $F(G)$ .

# The Fundamental Concept of Tikhonov

This concept consists of the following three conditions which should be in place when solving the ill-posed problem (2):

1. One should *a priori* assume that **there exists an ideal exact solution  $x^*$**  of equation (2) for an ideal noiseless data  $y^*$ .
2. The correctness set  $G$  should be chosen ***a priori***, meaning that some *a priori* bounds imposed on the solution  $x$  of equation (2) should be imposed.
3. To construct a stable numerical method for the problem (2), one should **assume** that there exists a family  $\{y_\delta\}$  of right hand sides of equation (2), where  $\delta > 0$  is the level of the error in the data with  $\|y^* - y_\delta\|_{B_2} \leq \delta$ . Next, one should construct a family of approximate solutions  $\{x_\delta\}$  of equation (2), where  $x_\delta$  corresponds to  $y_\delta$ . The family  $\{x_\delta\}$  should be such that

$$\lim_{\delta \rightarrow 0^+} \|x_\delta - x^*\| = 0.$$

# Quasi-solution

Another approach to the solution of ill-posed problem is concept of quasi-solution. This concept was introduced by Ivanov in 1962 in the work [Ivanov, 1962]. Let  $A$  be a compact operator,  $x \in M$ ,  $M$  is a compact set such that  $M \subset Q$ ,  $f \in A(M) \subset F$ : Then approximate solution of the problem

$$Ax = f$$

can be obtained by

$$x = A^{-1}f.$$

for small perturbations in the rhs  $f$ .

The main point here is that  $f \in A(M) \subset F$  otherwise the solution can not be obtained by  $x = A^{-1}f$ . Since it is difficult to check if  $f \in A(M) \subset F$  then it was introduced the concept of quasi-solution.

V.K.Ivanov, On linear problems which are not well-posed, Dokl.Akad.Nauk SSSR, 145(2), 211-223.  
1962 (In Russian)

# Quasi-solution

**Definition** (Ivanov, 1962)

A **quasi-solution** to the equation

$$Ax = f \quad (3)$$

on a set  $M \subset Q$  is an element  $x_K \in M$  that minimizes the residual

$$R(Ax_k, f) = \inf_{x \in M} R(Ax, f) \quad (4)$$

If  $M$  is a compact set then there exists a quasi-solution for any  $f \in F$ .  
If in addition  $f \in A(M)$  the quasi-solutions  $x_k$  (it can be a lot of such solutions) are the same as exact solution  $x$ .

Here is a sufficient condition for a quasi-solution to be unique and continuously depend on the rhs  $f$ .

# Quasi-solution

- **Theorem** [Ivanov, 2002, Tikhonov Arsenin, 1974]  
Assume that the equation (3) has at most one solution on a compact set  $M$  and  $\forall f \in M$  the projection  $Pf$  into  $A(M)$  is unique. Then a quasi-solution of the equation (3) is unique and continuously depends on  $f$ .
- We can conclude that the problem of finding a quasi-solution on a compact set is well-posed problem.
- If the quasi-solution is not unique, then its quasi-solutions form a subset of the compact set  $M$  and in this case this set continuously depends on  $f$  (Ivanov, 1963).

V. K. Ivanov, V. V. Vasin, V. P. Tanana, *Theory of linear ill-posed problems and its applications*, VSP, Utrecht, 2002.

A. N. Tikhonov, V. Ya. Arsenin, *Solutions of ill-posed problems*, Wiley, 1977.

# Ill-posed problem: differentiation of a function given with a noise

Suppose that the function  $f(x)$ ,  $x \in [0, 1]$  is given with a noise, i.e. suppose that instead of  $f(x) \in C^1[0, 1]$  the following function  $f_\delta(x)$  is given

$$f_\delta(x) = f(x) + \delta f(x), x \in [0, 1],$$

where  $\delta f(x)$  is the noisy component. Let  $\delta > 0$  be a small parameter such that  $\|\delta f\|_{C[0,1]} \leq \delta$ . Let us show that the problem of calculating the derivative  $f'_\delta(x)$  is unstable.

# Examples of ill-posed problems. Differentiation of a function given with a noise

For example, take

$$\delta f(x) = \frac{\sin(n^2 x)}{n},$$

where  $n > 0$  is a large integer. Then the  $C[0, 1]$ -norm of the noisy component is small,

$$\|\delta f\|_{C[0,1]} \leq \frac{1}{n}.$$

However, the difference between derivatives of noisy and exact functions

$$f'_\delta(x) - f'(x) = \delta f'(x) = n \cos n^2 x$$

is not small in any reasonable norm.

# Ill-posed problem: differentiation of a function given with a noise

A simple regularization method of stable calculation of derivatives is that the step size  $h$  in the corresponding finite difference discretization should be connected with the level of noise  $\delta$ .

$$f'_\delta(x) \approx \frac{f(x+h) - f(x)}{h} + \frac{\delta f(x+h) - \delta f(x)}{h}. \quad (5)$$

The first term in the right hand side of (5) is close to the exact derivative  $f'(x)$ , if  $h$  is small enough. The second term, however, comes from the noise and we need to balance these two terms via an appropriate choice of  $h = h(\delta)$ .

$$\left| f'_\delta(x) - \frac{f(x+h) - f(x)}{h} \right| = \left| \frac{\delta f(x+h) - \delta f(x)}{h} \right| \leq \frac{2\delta}{h}.$$

Hence, we should choose  $h = h(\delta)$  such that

$$\lim_{\delta \rightarrow 0} \frac{2\delta}{h(\delta)} = 0.$$

# Ill-posed problem: differentiation of a function given with a noise

For example, let  $h(\delta) = \delta^\mu$ , where  $\mu \in (0, 1)$ . Then

$$\lim_{\delta \rightarrow 0} \left| f'_\delta(x) - \frac{f(x+h) - f(x)}{h} \right| \leq \frac{2\delta}{h} = \frac{2\delta}{\delta^\mu} \leq \lim_{\delta \rightarrow 0} (2\delta^{1-\mu}) = 0.$$

Hence, the problem becomes stable for this choice of the grid step size  $h(\delta) = \delta^\mu$ . This means that  $h(\delta)$  is the regularization parameter.

# Ill-posed problem: integral equation of the first kind

- Let  $\Omega \subset \mathbb{R}^n$  is a bounded domain and the function  $K(x, y) \in C(\overline{\Omega} \times \overline{\Omega})$ . Recall that the equation

$$g(x) + \int_{\Omega} K(x, y)f(y)dy = f(x), x \in \Omega, \quad (6)$$

is called *integral equation of the second kind*. The problem is to find function  $f(x)$  by known  $K(x, y)$  and  $g(x)$ . These equations are considered quite often in the classic theory of PDEs and are solved by Liouville-Neumann (iterative) series.

- Next, let  $\Omega' \subset \mathbb{R}^n$  be a bounded domain and the function  $K(x, y) \in C(\overline{\Omega} \times \overline{\Omega})$ . Unlike (6), the equation

$$\int_{\Omega} K(x, y)f(y)dy = p(x), x \in \Omega' \quad (7)$$

is called the integral equation of the first kind. The Fredholm theory does not work for such equations. The problem of solution of equation (7) is an ill-posed problem.

## Ill-posed problem: integral equation of the first kind

Consider equation (7):

$$\int_{\Omega} K(x, y) f(y) dy = p(x), x \in \Omega'$$

The function  $K(x, y)$  is called *kernel* of the integral operator. Equation (7) can be rewritten in the form

$$Af = p, \tag{8}$$

where  $A : C(\overline{\Omega}) \rightarrow C(\overline{\Omega}')$  is the integral operator in (7). It is well known from the standard Functional Analysis course that  $A$  is a compact operator. We now show that the problem (8) is an ill-posed problem.

# Example of an integral equation of the first kind

Let  $\Omega = (0, 1)$ ,  $\Omega' = (a, b)$ . Let  $f_n(x) = f(x) + \sin nx$ . Then for  $x \in (0, 1)$

$$\int_0^1 K(x, y) f_n(y) dy = \int_0^1 K(x, y) f(y) dy + \int_0^1 K(x, y) \sin ny dy = g_n(x), \quad (9)$$

where  $g_n(x) = p(x) + p_n(x)$  and

$$p_n(x) = \int_0^1 K(x, y) \sin ny dy.$$

By the Lebesgue lemma

$$\lim_{n \rightarrow \infty} \|p_n\|_{C[a,b]} = 0.$$

However, it is clear that

$$\|f_n(x) - f(x)\|_{C[0,1]} = \|\sin nx\|_{C[0,1]}$$

is not small for large  $n$ .

## Tikhonov's theorem

**Theorem** (Tikhonov, 1943). *Let  $B_1$  and  $B_2$  be two Banach spaces. Let  $U \subset B_1$  be a compact set and  $F : U \rightarrow B_2$  be a continuous operator. Assume that the operator  $F$  is one-to-one. Let  $V = F(U)$ . Then the inverse operator  $F^{-1} : V \rightarrow U$  is continuous.*

**Proof.** Assume the opposite: that the operator  $F^{-1}$  is not continuous on the set  $V$ . Then there exists a point  $y_0 \in V$  and a number  $\varepsilon > 0$  such that for any  $\delta > 0$  there exists a point  $y_\delta$  such that although  $\|y_\delta - y_0\|_{B_2} < \delta$ , still  $\|F^{-1}(y_\delta) - F^{-1}(y_0)\|_{B_1} \geq \varepsilon$ . Hence, there exists a sequence  $\{\delta_n\}_{n=1}^\infty$ ,  $\lim_{n \rightarrow \infty} \delta_n = 0^+$  and the corresponding sequence  $\{y_n\}_{n=1}^\infty \subset V$  such that

$$\|y_{\delta_n} - y_0\|_{B_2} < \delta_n, \quad \|\underbrace{F^{-1}(y_n)}_{x_n} - \underbrace{F^{-1}(y_0)}_{x_0}\|_{B_1} \geq \varepsilon. \quad (10)$$

Denote

$$x_n = F^{-1}(y_n), x_0 = F^{-1}(y_0). \quad (11)$$

Then by (10) we have

$$\|x_n - x_0\|_{B_1} \geq \varepsilon. \quad (12)$$

Since  $U$  is a compact set and all points  $x_n \in U$ , then one can extract a convergent subsequence  $\{x_{n_k}\}_{k=1}^{\infty} \subseteq \{x_n\}_{n=1}^{\infty}$  from the sequence  $\{x_n\}_{n=1}^{\infty}$ . Let  $\lim_{k \rightarrow \infty} x_{n_k} = \bar{x}$ . Then  $\bar{x} \in U$ . Since  $F(x_{n_k}) = y_{n_k}$  and the operator  $F$  is continuous, then by (10) and (11) we have:

$$\begin{aligned} x_n = F^{-1}(y_n) &\Rightarrow F(x_n) = y_n, \\ x_0 = F^{-1}(y_0) &\Rightarrow F(x_0) = y_0; \\ F(\bar{x}) = \lim_{k \rightarrow \infty} F(x_{n_k}) &= \lim_{k \rightarrow \infty} y_{n_k} = y_0 \end{aligned} \quad (13)$$

So, we obtained,  $F(\bar{x}) = y_0$  since we have that

$$\|y_{\delta_n} - y_0\|_{B_2} < \delta_n$$

But also  $F(x_0) = y_0$ . Since the operator  $F$  is one-to one, we should have  $\bar{x} = x_0$ . However, by (12)  $\|\bar{x} - x_0\|_{B_1} \geq \varepsilon$ . We got a contradiction.  $\square$

# Model inverse problems

We will discuss following model inverse problems:

- Elliptic inverse problems
  - Elliptic CIPs
  - Cauchy problem
  - Inverse source problem
  - Inverse spectral problem
- Hyperbolic CIPs
- Parabolic CIPs
- Determination of the initial condition in hyperbolic or parabolic PDE

# Inverse source problem

The classical linear inverse problem is to recover a source function  $f$  in the equation

$$-\Delta u = f, \quad x \in \Omega \quad (14)$$

from the Cauchy data  $(g, h)$  on the boundary  $\Gamma$ :

$$u = g, \quad x \in \Gamma, \quad (15)$$

$$\frac{\partial u}{\partial n} = h, \quad x \in \Gamma. \quad (16)$$

Applications of this problem are in electroencephalography to determine electrical activities of brain from electrodes placed on a head, and electrocardiography to determine heart's electrical activity from body-surface potential distribution. This problem doesn't have a unique solution.

This can be proved if we add one compactly supported function and obtain a different source on the rhs of (14) with the same Cauchy data which is not changed.

## Inverse source problem: example of non-uniqueness

Let  $\Omega \subset \mathbb{R}^d$ ,  $d = 1, 2, 3$  is a bounded domain with the boundary  $\Gamma$ . Let  $\omega_i$ ,  $i = 1, 2$  be two balls which have different radius  $r_i$ , respectively, centered at the origin  $o$ , and these balls are inside the domain  $\Omega$ . Choose the scalars  $\lambda_i : \lambda_1 r_1^d = \lambda_2 r_2^d$ . Let the source has the form  $f_i = \lambda_i \xi_{\omega_i}$ , where  $\xi$  denotes the characteristic, or indicator function of the set  $S$  in the Laplace equation for  $i = 1, 2$

$$-\Delta u_i = f_i, \quad x \in \Omega, \quad (17)$$

$$u_i = g, \quad x \in \Gamma. \quad (18)$$

## Inverse source problem: example of non-uniqueness

Then for  $\forall v \in H(\Omega) = \{v \in H^2(\Omega) : \Delta v = 0\}$  the variational formulation of (17) will be:

$$-(\Delta u_i, v)_\Omega = -[(v, \frac{\partial u_i}{\partial n})_\Gamma - (\nabla u_i, \nabla v)_\Omega] \quad (19)$$

$$= -(v, \frac{\partial u_i}{\partial n})_\Gamma + (u_i, \frac{\partial v}{\partial n})_\Gamma - (u_i, \Delta v)_\Omega \quad (20)$$

$$= (u_i, \frac{\partial v}{\partial n})_\Gamma - (v, \frac{\partial u_i}{\partial n})_\Gamma = (f_i, v), \quad (21)$$

where  $(\cdot, \cdot)$  is the standard  $L_2$  inner product.

## Inverse source problem: example of non-uniqueness

Since  $u_i = g, i = 1, 2, x \in \Gamma$  then the equation above can be rewritten as

$$(f_i, v)_\Omega = (g, \frac{\partial v}{\partial n})_\Gamma - (v, \frac{\partial u_i}{\partial n})_\Gamma, i = 1, 2. \quad (22)$$

By the mean value theorem for harmonic functions we have

$$(f_i, v)_\Omega = |\omega_i|v(O), \quad i = 1, 2. \quad (23)$$

By construction of  $f_i$  and using (22), (23) we get

$$\forall v \in H(\Omega) = \{v \in H^2(\Omega) : \Delta v = 0\}$$

$$0 = (f_2, v) - (f_1, v) = (v, \frac{\partial u_1}{\partial n})_\Gamma - (v, \frac{\partial u_2}{\partial n})_\Gamma, i = 1, 2. \quad (24)$$

Since  $\forall h \in H^{1/2}(\Gamma)$  there exists the harmonic function  $v \in H(\Omega)$  such that  $v = h$  on  $\Gamma$  and  $H^{1/2}$  is dense in  $L^2(\Gamma)$  we conclude that

$(\frac{\partial u_1}{\partial n})_\Gamma = (\frac{\partial u_2}{\partial n})_\Gamma$  what means that two different sources have identical Cauchy data.

# How to solve inverse source problems

- In practical applications it is often required minimum-norm sources or harmonic sources such that  $\Delta f = 0$
- Often are considered localized sources modeled by monopoles, dipoles or their combinations. In the case of combinations of monopoles and dipoles can be obtained unique recovery of the source function via Holmgren's theorem.
- Direct algorithms for location of monopoles and dipoles are developed in  
El Badia, Ha-Duong, An inverse source problem in potential analysis, *Inverse Problems*, 16, pp.651–663, 2000.

# Inverse spectral problem

The forward problem is

$$Au = \lambda u, \quad (25)$$

where  $A$  is an elliptic operator,  $\lambda$  is eigenvalue and  $u$  is respective eigenfunction.

The **inverse spectral problem** is to recover the coefficients in the operator  $A$  or the geometry of the domain  $\Omega$  from partial or multiple spectral data (knowledge of eigenvalues and eigenfunctions).

# Example of an inverse spectral problem

Let the operator  $A$  is applied to  $u$  as:

$$Au = -u''(t) + q(t)u(t), \quad t \in (0, 1) \quad (26)$$

where  $q(t)$  is the potential. Then the classical **Sturm-Liouville problem** reads: given a potential  $q(t)$  and constants  $h, H > 0$  find eigenvalues  $\{\lambda_k\}$  and eigenfunctions  $\{u_k\}$  such that

$$-u''(t) + q(t)u(t) = \lambda u(t), \quad t \in (0, 1), \quad (27)$$

$$u'(0) - hu(0) = 0, \quad (28)$$

$$u'(1) + Hu(1) = 0. \quad (29)$$

The set of eigenvalues  $\{\lambda_k\}$  is real and countable.

The **inverse Sturm-Liouville problem** is to recover the potential  $q(t)$ ,  $h, H$  from the knowledge of spectral data  $(\{\lambda_k\}, \{u_k\})$ . This data can take different forms.

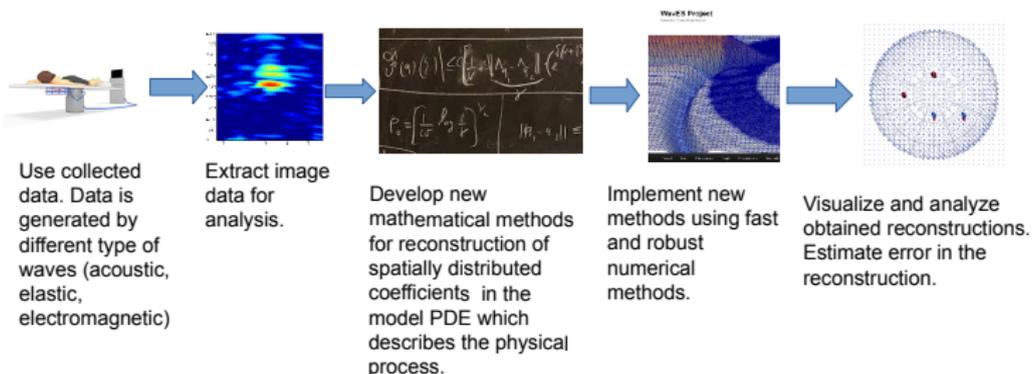
Numerical solution of these problems is presented in

M. T. Chu, G. H. Golub, *Inverse eigenvalue problems*, Oxford University Press, Oxford, New York, 2005.

# Coefficient Inverse Problems: main steps in solution

## Coefficient Inverse Problems for PDE

A coefficient inverse problem for a given partial differential equation (PDE) aims at estimating a spatially distributed coefficient of the model PDE using measurements taken on the boundary of the domain of interest.



# The Hyperbolic Coefficient Inverse Problem

Let us assume that the domain  $\Omega$  is a ball,  
 $\Omega = \{|x| < R\} \subset \mathbb{R}^n, R = \text{const.} > 0$ . Let  $T = \text{const.} > 0$ . Denote  
 $Q_T^\pm = \Omega \times (-T, T), S_T^\pm = \partial\Omega \times (-T, T)$ .  
 Let the function  $u(x, t) \in C^2(\overline{Q_T})$  satisfies to the

$$c(x)u_{tt} = \Delta u + \sum_{|\alpha| \leq 1} a_\alpha(x) D_x^\alpha u, \quad \text{in } Q_T, \quad (30)$$

$$u(x, 0) = f_0(x), \quad u_t(x, 0) = f_1(x), \quad (31)$$

$$u|_{S_T} = p(x, t), \quad \frac{\partial u}{\partial n}|_{S_T} = q(x, t), \quad (32)$$

where functions  $a_\alpha, c \in C(\overline{Q_T})$  and  $c \geq 1$ .

**The Hyperbolic Coefficient Inverse Problem.** Suppose that one of coefficients in equation (30) is unknown inside of the ball  $\Omega$  and is known outside of it. Assume that all other coefficients in (33) are known and conditions (31) are satisfied. Determine that unknown coefficient inside of  $\Omega$ , assuming that the functions  $p(x, t)$  and  $q(x, t)$  in (32) are known.

# The Coefficient Inverse Problem for a parabolic equation

Consider the Cauchy problem for the following forward parabolic equation

$$c(x)u_t = \Delta u + \sum_{|\alpha| \leq 1} a_\alpha(x) D_x^\alpha u, \text{ in } D_T^{n+1} = \mathbb{R}^n \times (0, T), \quad (33)$$

$$u(x, 0) = f_0(x), \quad (34)$$

$$c, a_\alpha \in C^\beta(\mathbb{R}^n), f_0 \in C^{2+\beta}(\mathbb{R}^n), \beta \in (0, 1), c(x) \geq 1. \quad (35)$$

Given conditions (35), this problem has unique solution

$u \in C^{2+\beta, 1+\beta/2}(\overline{D}_T^{n+1})$ . Assume that  $\Omega = \{|x| < R\} \subset \mathbb{R}^n, n \geq 2$ . Let  $\Gamma \subseteq \partial\Omega$  be a part of the boundary of the domain  $\Omega$  and  $T = \text{const.} > 0$ .

**The Parabolic Coefficient Inverse Problem.** Suppose that one of coefficients in equation (33) is unknown inside of the ball  $\Omega$  and is known outside of it. Assume that all other coefficients in (33) are known and conditions (34), (35) are satisfied. Determine that unknown coefficient inside of  $\Omega$ , assuming that the following functions  $p(x, t)$  and  $q(x, t)$  are known

$$u|_{\Gamma_T} = p(x, t), \quad \frac{\partial u}{\partial n}|_{\Gamma_T} = q(x, t). \quad (36)$$

## CIP for a parabolic equation is an ill-posed problem

Let the function  $a(x) \in C^\alpha(\mathbb{R}^n)$ ,  $\alpha \in (0, 1)$  and  $a(x) = 0$  outside of the bounded domain  $\Omega \subset \mathbb{R}^n$  with  $\partial\Omega \in C^3$ . Consider the following Cauchy problem

$$u_t = \Delta u + a(x)u, \quad (x, t) \in D_T^{n+1}, \quad (37)$$

$$u(x, 0) = f(x). \quad (38)$$

Here the function  $f(x) \in C^{2+\alpha}(\mathbb{R}^n)$  has a finite support in  $\mathbb{R}^n$ . Another option for the initial condition is

$$f(x) = \delta(x - x_0), \quad x_0 \notin \bar{\Omega} \quad (39)$$

The inverse problem is: assume that the function  $a(x)$  is unknown inside of the domain  $\Omega$ . Determine this function for  $x \in \Omega$  assuming that the following function  $g(x, t)$  is known

$$u|_{S_T} = g(x, t). \quad (40)$$

# CIP for a parabolic equation is an ill-posed problem

Let us show that this CIP is an ill-posed problem. Let the function  $u_0$  be the fundamental solution of the heat equation  $u_{0t} = \Delta u_0$ ,

$$u_0(x, t) = \frac{1}{(2\sqrt{\pi t})^n} \exp\left(-\frac{|x|^2}{4t}\right).$$

It is well known that by [LSU]

$$u(x, t) = \int_{\mathbb{R}^n} u_0(x - \xi, t) f(\xi) d\xi + \int_0^t \int_{\Omega} u_0(x - \xi, t - \tau) a(\xi) u(\xi, \tau) d\tau. \quad (41)$$

Because of the presence of the integral  $\int_0^t (\cdot) d\tau$  the integral (41) is a Volterra-like integral equation of the second kind.

[LSU] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, AMS, Providence, R.I., 1968.

Hence, it can be solved as in [LSU]

$$u(x, t) = \underbrace{\int_{\mathbb{R}^n} u_0(x - \xi, t) f(\xi) d\xi}_{u_0^f} + \sum_{n=1}^{\infty} u_n(x, t), \quad (42)$$

$$u_n(x, t) = \int_0^t \int_{\Omega} u_0(x - \xi, t - \tau) a(\xi) u_{n-1}(\xi, \tau) d\tau.$$

One can prove that each function  $u_n \in C^{2+\alpha, 1+\alpha/2}(\overline{D_T^{n+1}})$  and using [LSU]

$$\left| D_x^\beta D_t^k u_n(x, t) \right| \leq \frac{(Mt)^n}{n!}, \quad |\beta| + 2k \leq 2, \quad (43)$$

where  $M = \|a\|_{C^\alpha(\overline{\Omega})}$ . In the case when  $f = \delta(x - x_0)$  the first term in the right hand side of (42) should be replaced with  $u_0(x - x_0, t)$ .

Let  $u_0^f(x, t)$  be the first term of the right hand side of (42) and  $v(x, t) = u(x, t) - u_0^f(x, t)$ .

Using (43), one can rewrite (42) as

$$v(x, t) = \int_0^t \int_{\Omega} u_0(x - \xi, t - \tau) (a(\xi) u_0^f(\xi, \tau) + P(a)(\xi, \tau)) d\xi d\tau, \quad (44)$$

where  $P(a)$  is a nonlinear operator applied to the function  $a$ . It is clear from (42)-(44) that the operator  $P : C^\alpha(\overline{\Omega}) \rightarrow C^{2+\alpha, 1+\alpha/2}(\overline{Q_T})$  is continuous. Setting in (44)  $(x, t) \in S_T$ , recalling (40) and denoting  $\bar{g}(x, t) = g(x, t) - u_0^f(x, t)$ , we obtain a nonlinear integral equation of the first kind with respect to the unknown coefficient  $a(x)$

$$\underbrace{\int_{S_T} u_0(x - \xi, t - \tau) (u_0^f(\xi, \tau) a(\xi) + P(a)(\xi, \tau)) d\xi d\tau}_{A(a)} = \bar{g}(x, t), \quad (x, t) \in S_T. \quad (45)$$

Let  $A(a)$  be the operator in the left hand side of (45). Let  $H_1 = L_2(\Omega)$  and  $H_2 = L_2(S_T)$ . Consider now the set  $U$  of functions defined as

$$U = \left\{ a : a \in C^\alpha(\overline{\Omega}), \|a\|_{C^\alpha(\overline{\Omega})} \leq M \right\} \subset H_1.$$

Since the  $L_2(\Omega)$ -norm is weaker than the  $C^\alpha(\overline{\Omega})$ -norm, then  $U$  is a bounded set in  $H_1$  and  $A : U \rightarrow C(S_T)$  is a compact operator by Theorem 1.1 of [BK]. Since the norm in  $L_2(S_T)$  is weaker than the norm in  $C(S_T)$ , then  $A : U \rightarrow H_2$  is also a compact operator. However,  $U$  is not a compact. Hence, from the Theorem about an ill-posed problem (Theorem 1.2 of [BK]) follows that the problem of solution of the equation

$$A(a) = g, a \in U \subset H_1, g \in H_2$$

is an ill-posed problem.

# Determination of initial condition in parabolic PDE

- The inverse heat conduction problem has applications in image processing, remote sensing, oil base detection etc.
- Consider the direct Cauchy problem for the following parabolic equation

$$\begin{aligned}u_t &= C^2(x)\Delta u, \\u(\cdot, 0) &= f(x).\end{aligned}\tag{46}$$

If (46) is considered in  $\Omega \times (0, T]$  and supplied by appropriate boundary conditions, then with known  $f(x) \in C^2(\mathbb{R}^n)$ , boundary conditions and  $C(x) \in C(\mathbb{R}^n)$  the problem can be solved.

- **The inverse problem is:** determine the initial condition  $f(x)$  in (46) by knowing the function  $u$  at the final time  $T$ :

$$u(\cdot, T) = g(x).\tag{47}$$

# Determination of initial condition in parabolic PDE

Two types of algorithms can be considered to solve the inverse problem:

- Optimization approach and construction of the adjoint problem to be used in the iterative gradient update. Read about this approach in

Dinh Nho Hào Nguyen Thi Ngoc Oanh, Determination of the initial condition in parabolic equations from integral observations, *Inverse Problems in Science and Engineering*, 25:8, 1138-1167, 2017. DOI: 10.1080/17415977.2016.1229778

- Algorithm for solution of the inverse problem which is based on the reduction of solution of (46) to the solution of Fredholm integral equation of the first kind by the method of separation of variables and then obtaining of the initial condition  $f(x)$  by the inverse Fourier transform of this solution. See more info in

Tao Min, Bei Geng, Jucheng Ren, Inverse estimation of the initial condition for the heat equation, *International Journal of Pure and Applied Mathematics*, 82(4), 581-593, 2013.

## Acoustic CIPs

- **Acoustic CIPs** for acoustic wave equation

$$\frac{1}{c^2(x)} u_{tt} = \Delta u \text{ in } \mathbb{R}^3 \times (0, \infty), \quad (48)$$

$$u(x, 0) = 0, u_t(x, 0) = \delta(x - x_0). \quad (49)$$

- Let now introduce the convex bounded domain  $\Omega \subset \mathbb{R}^3$  with the boundary  $\partial\Omega \in C^3$  and specify time variable  $t \in [0, T]$ . Next, we supply the Cauchy problem by the appropriate b.c.
- We assume that the coefficient  $c(x)$  belongs to the set of admissible parameters  $M$  which should be specified for the concrete problem.
- $u(x, t)$  acoustic pressure - we measure it on the boundary  $\partial\Omega$ .
- $c(x)$  speed of sound – want to determine by measured  $u(x, t)$  on the boundary  $\partial\Omega$
- Applications: medical imaging, electromagnetic, acoustics, geological profiling, construction of new materials

# Acoustic CIP: example

We model the process of electric wave field propagation in non-conductive and nonmagnetic media with  $\nabla \cdot (\varepsilon E) = 0$  via a single hyperbolic PDE, which is the same as an acoustic wave equation (48)-(49). The forward problem is the following Cauchy problem

$$\varepsilon_r(x)u_{tt} = \Delta u, \text{ in } \mathbb{R}^3 \times (0, \infty), \quad (50)$$

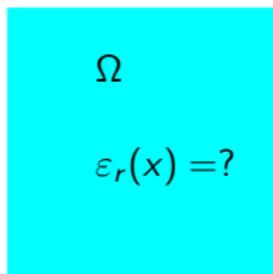
$$u(x, 0) = 0, \quad u_t(x, 0) = \delta(x - x_0). \quad (51)$$

Here,  $\varepsilon_r(x)$  is the spatially distributed dielectric constant (relative dielectric permittivity),

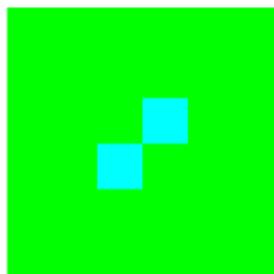
$$\varepsilon_r(x) = \frac{\varepsilon(x)}{\varepsilon_0}, \quad \sqrt{\varepsilon_r(x)} = n(x) = \frac{c_0}{c(x)} \geq 1, \quad (52)$$

where  $\varepsilon_0$  is the dielectric permittivity of the vacuum (which we assume to be the same as the one in the air),  $\varepsilon(x)$  is the dielectric permittivity of the medium of interest,  $n(x)$  is the refractive index of the medium of interest,  $c(x)$  is the speed of the propagation of the EM field in this medium, and  $c_0$  is the speed of light in the vacuum, which we assume to be the same as one in the air.

## Acoustic CIP: example



$$u(x, t) = g(x, t) \text{ on } \partial\Omega$$

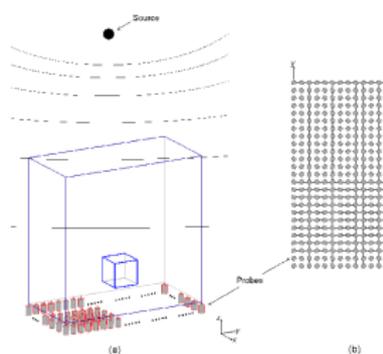


$$\text{exact } \varepsilon_r(x)$$

**Coefficient Inverse Problem** Assume that the function  $\varepsilon_r(x)$  is unknown in the domain  $\Omega$ . Determine the function  $\varepsilon_r(x)$  for  $x \in \Omega$ , assuming that the following function  $g(x, t)$  is known for a source  $x_0 \notin \bar{\Omega}$

$$u(x, t) = g(x, t), \forall (x, t) \in \partial\Omega \times (0, T].$$

## Reconstruction of dielectrics from experimental data



a) The rectangular prism depicts our computational domain  $\Omega$ . Only a single source location outside of this prism was used. Tomographic measurements of the scattered time resolved EM wave were conducted on the bottom side of this prism. The signal was measured with the time interval 20 picoseconds with total time 12.3 nanoseconds. b) Schematic diagram of locations of detectors on the bottom side of the prism  $\Omega$ . The distance between neighboring detectors was 10 mm.

L.Beilina, M.V.Klibanov, Reconstruction of dielectrics from experimental data via a hybrid globally convergent/adaptive inverse algorithm, *Inverse Problems*, 26, 125009, 2010.

# The two-stage numerical procedure for solution of CIP

**Stage 1.** Approximately globally convergent numerical method provides a good approximation for the exact solution  $\varepsilon_{glob}$ .

**Stage 2.** Adaptive Finite Element Method refines it via minimization of the corresponding Tikhonov functional with  $\varepsilon_0 = \varepsilon_{glob}$ :

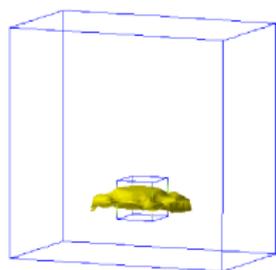
$$J(u, \varepsilon) = \frac{1}{2} \int_{\Gamma} \int_0^T (u - \tilde{u})^2 z_{\delta}(t) ds dt + \frac{1}{2} \gamma \int_{\Omega} (\varepsilon - \varepsilon_0)^2 dx. \quad (53)$$

where  $\tilde{u}$  is the observed wave field in the model PDE (for example, acoustic wave equation),  $u$  satisfies this model PDE and thus depends on  $\varepsilon$ .

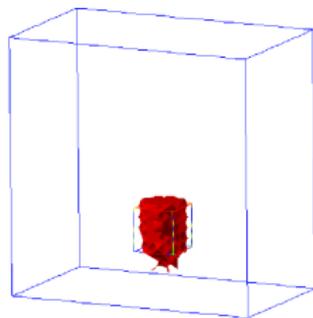
# The two-stage numerical procedure

**Stage 1.** Approximately globally convergent numerical method provides a good approximation for the exact solution.

**Stage 2.** Adaptive Finite Element Method refines it.



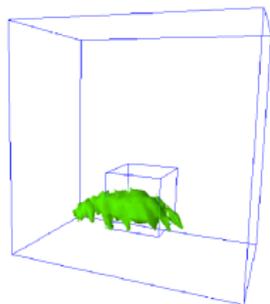
$$\text{a) } \varepsilon_r^{(5,2)} = 3.9, n^{(5,2)} = 1.97$$



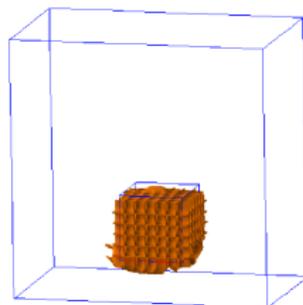
$$\text{b) } \varepsilon_{r,h} \approx 4.2, n_{glob} = \sqrt{\varepsilon_{r,h}} \approx 2.05$$

a) A sample of the reconstruction result of the dielectric cube No. 1 (4 cm side) via the first stage. b) Result after applying the adaptive stage (2-nd stage). The side of the cube is 4 cm=1.33 wavelength.

## Results of the two-stage procedure, cube nr.2 (big)



a)  $\varepsilon_r(5, 5) = 3.19, n^{(5,5)} = 1.79$



b)  $\varepsilon_{r,h} \approx 3.0, n_{glob} = \sqrt{\varepsilon_{r,h}} \approx 1.73$

a) Reconstruction of the dielectric cube No. 2 (6 cm side) via the first stage. b) The final reconstruction result after applying the adaptive stage (2-nd stage). The side 6 cm=2 wavelength.

L.Beilina, M.V.Klibanov, Reconstruction of dielectrics from experimental data via a hybrid globally convergent/adaptive inverse algorithm, *Inverse Problems*, 26, 125009, 2010.

## Elastic CIPs

Let  $v(x, t) = (v_1, v_2, v_3)(x, t)$  be vector of displacement. We consider the Cauchy problem for the elastodynamics equations in the isotropic case in the entire space  $\mathbb{R}^3$ ,

$$\begin{aligned} \rho(x) \frac{\partial^2 v}{\partial t^2} - \nabla \cdot \tau &= \delta(x_3 - z_0) f(t), \\ \tau &= C \epsilon, \\ v(x, 0) = 0, v_t(x, 0) &= 0, \quad x \in \mathbb{R}^3, t \in (0, T), \end{aligned} \tag{54}$$

where  $v(x, t)$  is the total displacement generated by the incident plane wave  $f(t)$  propagating along the  $x_3$ -axis which is incident at the plane  $x_3 = z_0$ ,  $\rho(x)$  is the density of the material,  $\tau$  is the stress tensor,  $C$  is a cyclic symmetric tensor and  $\epsilon$  is the strain tensor which have components

$$\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \quad i, j = 1, 2, 3.$$

The strain tensor  $\epsilon$  is coupled with the stress tensor  $\tau$  by the Hooke's law

$$\tau_{i,j} = \sum_{k=1}^3 \sum_{l=1}^3 C_{ijkl} \epsilon_{kl} \tag{55}$$

## Elastic CIPs

$C$  is a cyclic symmetric tensor

$$C_{ijkl} = C_{klij} = C_{jkli}.$$

When  $C_{ijkl}$  does not depend on  $\mathbf{x}$  then material of the domain which we consider is said to be **homogeneous**. If the tensor  $C_{ijkl}$  does not depend on the choice of the coordinate system, then the material of the domain under interest is said to be **isotropic**. Otherwise, the material is **anisotropic**.

In the **isotropic** case the cyclic symmetric tensor  $C$  can be written as

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk}),$$

where  $\delta_{ij}$  is the Kronecker delta, in which case the equation (55) takes the form

$$\tau_{i,j} = \lambda \delta_{ij} \sum_{k=1}^3 \epsilon_{kk} + 2\mu \epsilon_{ij}, \quad (56)$$

where  $\lambda$  and  $\mu$  are **Lame coefficients**.

## Elastic CIPs

Lame coefficients  $\lambda$  and  $\mu$  are given by

$$\begin{aligned}\mu &= \frac{E}{2(1+\nu)}, \\ \lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)}.\end{aligned}\tag{57}$$

Here,  $E$  is the **modulus of elasticity**, or **Young modulus**, and  $\nu$  is the **Poisson's ratio** of the elastic material. Following relations should be satisfied

$$\lambda > 0, \mu > 0 \iff E > 0, 0 < \nu < 1/2.\tag{58}$$

## Elastic CIPs

To write the equation (54) only in terms of  $v$  we eliminate the strain tensor  $\epsilon$  from (54) using (55). Then in the isotropic case the equation in (54) writes

$$\begin{aligned}
 \rho(x) \frac{\partial^2 v_1}{\partial t^2} - \frac{\partial}{\partial x_1} \left( (\lambda + 2\mu) \frac{\partial v_1}{\partial x_1} + \lambda \frac{\partial v_2}{\partial x_2} + \lambda \frac{\partial v_3}{\partial x_3} \right) \\
 - \frac{\partial}{\partial x_2} \left( \mu \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) \right) - \frac{\partial}{\partial x_3} \left( \mu \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right) \right) = 0, \\
 \rho(x) \frac{\partial^2 v_2}{\partial t^2} - \frac{\partial}{\partial x_2} \left( (\lambda + 2\mu) \frac{\partial v_2}{\partial x_2} + \lambda \frac{\partial v_1}{\partial x_1} + \lambda \frac{\partial v_3}{\partial x_3} \right) \\
 - \frac{\partial}{\partial x_1} \left( \mu \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) \right) - \frac{\partial}{\partial x_3} \left( \mu \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right) \right) = 0, \\
 \rho(x) \frac{\partial^2 v_3}{\partial t^2} - \frac{\partial}{\partial x_3} \left( (\lambda + 2\mu) \frac{\partial v_3}{\partial x_3} + \lambda \frac{\partial v_2}{\partial x_2} + \lambda \frac{\partial v_1}{\partial x_1} \right) \\
 - \frac{\partial}{\partial x_2} \left( \mu \left( \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} \right) \right) - \frac{\partial}{\partial x_1} \left( \mu \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right) \right) = \delta(x_3 - z_0) f(t),
 \end{aligned} \tag{59}$$

## Elastic CIPs

The system above for  $\lambda = \text{const.} > 0, \mu = \text{const.} > 0$  can be written in a more compact form as

$$\rho \frac{\partial^2 v}{\partial t^2} - \mu \nabla \cdot (\nabla v) - (\lambda + \mu) \nabla (\nabla \cdot v) = \delta(x_3 - z_0) f(t),$$

$$v(x, 0) = 0, v_t(x, 0) = 0, \quad x \in \mathbb{R}^3, t \in (0, T).$$
(60)

Inserting Helmholtz decomposition

$$v = \nabla \varphi + \nabla \times \psi \tag{61}$$

with a scalar potential  $\varphi$  and a vector potential  $\psi$  into (60) we get

$$\rho \frac{\partial^2 (\nabla \varphi + \nabla \times \psi)}{\partial t^2} - \mu \nabla \cdot (\nabla (\nabla \varphi + \nabla \times \psi)) - (\lambda + \mu) \nabla (\nabla \cdot (\nabla \varphi + \nabla \times \psi))$$

$$= \delta(x_3 - z_0) f(t)$$
(62)

## Elastic CIPs

Using

$$\begin{aligned}\nabla \cdot (\nabla \varphi) &= \Delta \varphi, \\ \nabla \cdot (\nabla \times \psi) &= 0,\end{aligned}$$

we finally get with  $f(t) = 0$

$$\nabla \left( \rho \frac{\partial^2 \varphi}{\partial t^2} - (\lambda + 2\mu) \Delta \varphi \right) + \nabla \times \left( \rho \frac{\partial^2 \psi}{\partial t^2} - \mu \Delta \psi \right) = 0 \quad (63)$$

We conclude that

$$\begin{aligned}\rho \frac{\partial^2 \varphi}{\partial t^2} - (\lambda + 2\mu) \Delta \varphi &= 0, \\ \rho \frac{\partial^2 \psi}{\partial t^2} - \mu \Delta \psi &= 0.\end{aligned} \quad (64)$$

Here,  $v = \nabla \varphi$  is the **pressure wave** with the speed  $V_p = \left(\frac{\lambda+2\mu}{\rho}\right)^{1/2}$ ,  
 $v = \nabla \times \psi$  is the **shear wave** with the speed  $V_s = \left(\frac{\mu}{\rho}\right)^{1/2}$

## Elastic CIP: examples of CIPs

 $\Omega$ 

$\rho(x) = ?$

$\lambda(x), \mu(x) \in \Omega$

$u(x, t) = g(x, t)$  on  $\partial\Omega$

 $\Omega$ 

$\rho(x) = ?$

$\lambda(x) = ?$

$\mu(x) = ?$

$v(x, t) = g(x, t)$  on  $\partial\Omega$

**Inverse Problem (IP1)** Determine the density function  $\rho(x)$  in  $\Omega$  for  $x \in \Omega$  assuming that the Lamé parameters  $\lambda(x), \mu(x)$  and  $g(x, t)$  s.t.

$$v(x, t) = g(x, t), \forall (x, t) \in \partial\Omega \times (0, T].$$

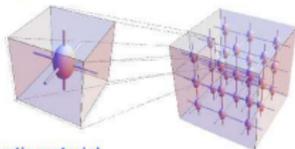
are known.

**Inverse Problem (IP2)** Determine the functions  $\rho(x), \lambda(x), \mu(x)$  in  $\Omega$  for  $x \in \Omega$  assuming that  $g(x, t)$  is known in  $\Omega$ .

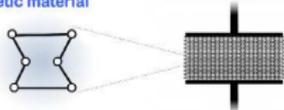
# Applications of elastic CIP: design of new materials

There is a class of materials for which the macroscale properties can be obtained more from such called mechanical *microstructural* design, see Figure for examples of such materials.

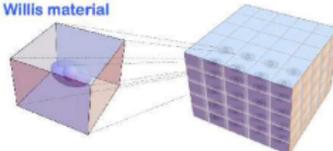
Micropolar material



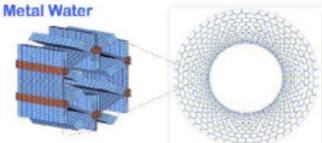
Auxetic material



Willis material



Metal Water



- Practical applications: mechanical cloaking, control and manipulation of waves in fluids and solids, etc.
- Examples of such materials include nanomaterials such as graphene or carbon nanotubes with extraordinary strength properties.
- Design of new mechanical metamaterials using computational modeling is one of the applications of elastic CIPs.

## CIP for electromagnetic problems. Maxwell's equations

Consider a region of space that has no electric or magnetic current sources, but may have materials that absorb electric or magnetic field energy. Then, using MKS units, the time-dependent Maxwell's equations are given in **differential** and **integral** form by *Faraday's law* :

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} - \mathbf{M} \quad (65a)$$

$$\frac{\partial}{\partial t} \iint_A \mathbf{B} \cdot d\mathbf{A} = -\oint_L \mathbf{E} \cdot d\mathbf{L} - \iint_A \mathbf{M} \cdot d\mathbf{A} \quad (65b)$$

*The MKS system of units is a physical system of units that expresses any given measurement using fundamental units of the metre, kilogram, and/or second (MKS))*

A. Taflove, S. C. Hagness, Computational Electromagnetics. The finite-difference time-domain method, 3rd edition, Artech House Publishers, 2005.

## Maxwell's equations

*Ampere's law :*

$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J} \quad (66a)$$

$$\frac{\partial}{\partial t} \iint_A \mathbf{D} \cdot d\mathbf{A} = \oint_L \mathbf{H} \cdot d\mathbf{L} - \iint_A \mathbf{J} \cdot d\mathbf{A} \quad (66b)$$

*Gauss' law for the electric field :*

$$\nabla \cdot \mathbf{D} = 0 \quad (67a)$$

$$\iint_A \mathbf{D} \cdot d\mathbf{A} = 0 \quad (67b)$$

*Gauss' law for the magnetic field :*

$$\nabla \cdot \mathbf{B} = 0 \quad (68a)$$

$$\iint_A \mathbf{B} \cdot d\mathbf{A} = 0 \quad (68b)$$

# Maxwell's equations

In (65) to (68), the following symbols (and their MKS units) are defined:

- E** : electric field (volts/meter)
- D** : electric flux density (coulombs/meter<sup>2</sup>)
- H** : magnetic field (amperes/meter)
- B** : magnetic flux density (webers/meter<sup>2</sup>)
- A** : arbitrary three-dimensional surface
- dA** : differential normal vector that characterizes surface  $A$  (meter<sup>2</sup>)
- L** : closed contour that bounds surface  $A$  (volts/meter)
- dL** : differential length vector that characterizes contour  $L$  (meters)
- J** : electric current density (amperes/meter<sup>2</sup>)
- M** : equivalent magnetic current density (volts/meter<sup>2</sup>)

# Maxwell's equations

In linear, isotropic, nondispersive materials (i.e. materials having field-independent, direction-independent, and frequency-independent electric and magnetic properties), we can relate  $\mathbf{D}$  to  $\mathbf{E}$  and  $\mathbf{B}$  to  $\mathbf{H}$  using simple proportions:

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_r \varepsilon_0 \mathbf{E}; \quad \mathbf{B} = \mu \mathbf{H} = \mu_r \mu_0 \mathbf{H} \quad (69)$$

where

- $\varepsilon$  : electrical permittivity (farads/meter)
- $\varepsilon_r$  : relative permittivity (dimensionless scalar)
- $\varepsilon_0$  : free-space permittivity ( $8.854 \times 10^{-12}$  farads/meter)
- $\mu$  : magnetic permeability (henrys/meter)
- $\mu_r$  : relative permeability (dimensionless scalar)
- $\mu_0$  : free-space permeability ( $4\pi \times 10^{-7}$  henrys/meter)

Note that  $\mathbf{J}$  and  $\mathbf{M}$  can act as *independent sources* of E- and H-field energy,  $\mathbf{J}_{source}$  and  $\mathbf{M}_{source}$ .

# Maxwell's equations

We also allow for materials with isotropic, nondispersive electric and magnetic losses that attenuate E- and H-fields via conversion to heat energy. This yields

$$\mathbf{J} = \mathbf{J}_{source} + \sigma \mathbf{E}; \quad \mathbf{M} = \mathbf{M}_{source} + \sigma^* \mathbf{H} \quad (70)$$

where  $\sigma$  : electric conductivity (siemens/meter)  
 $\sigma^*$  : equivalent magnetic loss (ohms/meter)

Finally, we substitute (69) and (70) into (65a) and (66a). This yields Maxwell's curl equations in linear, isotropic, nondispersive, lossy materials:

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \mathbf{E} - \frac{1}{\mu} (\mathbf{M}_{source} + \sigma^* \mathbf{H}) \quad (71)$$

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon} \nabla \times \mathbf{H} - \frac{1}{\varepsilon} (\mathbf{J}_{source} + \sigma \mathbf{E}) \quad (72)$$

## Maxwell's equations

We now write out the vector components of the curl operators of (71) and (72) in Cartesian coordinates. This yields the following system:

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu} \left[ \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - (M_{source_x} + \sigma^* H_x) \right] \quad (73a)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu} \left[ \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - (M_{source_y} + \sigma^* H_y) \right] \quad (73b)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu} \left[ \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - (M_{source_z} + \sigma^* H_z) \right] \quad (73c)$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon} \left[ \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - (J_{source_x} + \sigma E_x) \right] \quad (74a)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon} \left[ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - (J_{source_y} + \sigma E_y) \right] \quad (74b)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon} \left[ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - (J_{source_z} + \sigma E_z) \right] \quad (74c)$$

## CIPs for electric wave propagation

Recall Maxwell's curl equations in linear, isotropic, nondispersive, lossy materials with  $\sigma^* = 0$ ,  $\mathbf{M}_{source} = 0$ :

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \mathbf{E} \quad (75)$$

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon} \nabla \times \mathbf{H} - \frac{1}{\varepsilon} \sigma \mathbf{E} - \frac{1}{\varepsilon} \mathbf{J}_{source} \quad (76)$$

Take now  $\frac{\partial}{\partial t}$  from (76) and  $\nabla \times$  from (75) to get:

$$\nabla \times \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \frac{1}{\mu} \nabla \times \mathbf{E} \quad (77)$$

$$\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\partial}{\partial t} \nabla \times \mathbf{H} - \sigma \frac{\partial}{\partial t} \mathbf{E} - \frac{\partial}{\partial t} \mathbf{J}_{source} \quad (78)$$

## CIPs for electric wave propagation

Substitute the right hand side of (77) into (78) instead of  $\frac{\partial}{\partial t} \nabla \times \mathbf{H}$  to obtain Maxwell's equations for electric field  $\mathbf{E} = (E_1, E_2, E_3)$ . Let us consider now Cauchy problem for the Maxwell's equations for electric field  $\mathbf{E}$  in the domain  $\Omega_T = \Omega \times [0, T]$ :

$$\begin{aligned} \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \nabla \times \frac{1}{\mu} \nabla \times \mathbf{E} &= -\sigma \frac{\partial \mathbf{E}}{\partial t} - \frac{\partial \mathbf{J}_{source}}{\partial t} \text{ in } \Omega_T, \\ \nabla \cdot (\varepsilon \mathbf{E}) &= 0, \\ \mathbf{E}(\mathbf{x}, 0) = f_0(x), \quad \mathbf{E}_t(\mathbf{x}, 0) &= f_1(x) \text{ in } \Omega, \end{aligned} \tag{79}$$

- Let  $\Omega \subset \mathbb{R}^3$  be a convex bounded domain with the boundary  $\partial\Omega \in C^3$  and specify time variable  $t \in [0, T]$ . Next, we supply the Cauchy problem by the appropriate b.c.
- $\varepsilon(x)$  and  $\sigma(x)$  are dielectric permittivity and electric conductivity functions, respectively of the domain  $\Omega$ . In (79),  $\varepsilon(x) = \varepsilon_r(x)\varepsilon_0$ ,  $\mu = \mu_r\mu_0$  and  $\sigma(x)$  are dielectric permittivity, permeability and electric conductivity functions, respectively,  $\varepsilon_0, \mu_0$  are dielectric permittivity and permeability of free space,

## CIPs for electric wave propagation

 $\Omega$ 

$\varepsilon_r(x) = ?$

$\sigma = 0, \mu_r = 1$

$E(x, t) = g(x, t)$  on  $\partial\Omega$

 $\Omega$ 

$\varepsilon_r(x) = ?$   
 $\sigma(x) = ?$

$\mu_r \approx 1$

$E(x, t) = g(x, t)$  on  $\partial\Omega$

**Inverse Problem (EIP1)** Determine the relative dielectric permittivity function  $\varepsilon_r(x)$  in  $\Omega$  for  $x \in \Omega$  in nonconductive ( $\sigma(x) = 0$ ) and nonmagnetic ( $\mu_r = 1$ ) media when the measured function  $g(x, t)$  s.t.

$$\mathbf{E}(x, t) = g(x, t), \forall (x, t) \in \partial\Omega \times (0, T].$$

is known.

**Inverse Problem (EIP2)** Determine the functions  $\varepsilon(x), \sigma(x)$  in  $\Omega$  for  $x \in \Omega$  for  $\mu_r \approx 1$  in water assuming that  $g(x, t)$  is known in  $\partial\Omega \times (0, T]$ .

## CIPs for magnetic field

Similarly can be obtained Maxwell's equations for magnetic field  $\mathbf{H} = (H_1, H_2, H_3)$ . Let us consider system of Maxwell's equations in linear, isotropic, nondispersive, lossy materials with  $\sigma = 0, \sigma^* = 0$ :

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \mathbf{E} \quad (80)$$

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon} \nabla \times \mathbf{H} - \frac{1}{\varepsilon} \mathbf{J}_{source} \quad (81)$$

In this case we take time derivative in (80) and operator  $\nabla \times$  in (81) to get:

$$\frac{\partial^2 \mathbf{H}}{\partial t^2} = -\frac{1}{\mu} \frac{\partial}{\partial t} \nabla \times \mathbf{E}, \quad (82)$$

$$\nabla \times \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \frac{1}{\varepsilon} \nabla \times \mathbf{H} - \nabla \times \frac{1}{\varepsilon} \mathbf{J}_{source}. \quad (83)$$

Substitute the right hand side of (82) into (83) instead of  $\frac{\partial}{\partial t} \nabla \times \mathbf{E}$  to obtain Maxwell's equations for magnetic field  $\mathbf{H} = (H_1, H_2, H_3)$ :

$$\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} + \nabla \times \frac{1}{\varepsilon} \nabla \times \mathbf{H} = \nabla \times \frac{1}{\varepsilon} \mathbf{J}_{source} \quad (84)$$

## CIPs for magnetic field

Let us consider now Cauchy problem for magnetic field  $\mathbf{H}$  in the domain  $\Omega_T = \Omega \times [0, T]$ :

$$\begin{aligned} \mu \frac{\partial^2 \mathbf{H}}{\partial t^2} + \nabla \times \frac{1}{\varepsilon} \nabla \times \mathbf{H} &= \nabla \times \frac{1}{\varepsilon} \mathbf{J}_{source} \text{ in } \Omega_T, \\ \mathbf{H}(\mathbf{x}, 0) &= f_0(\mathbf{x}), \quad \mathbf{H}_t(\mathbf{x}, 0) = f_1(\mathbf{x}) \text{ in } \Omega, \end{aligned} \quad (85)$$

- Let  $\Omega \subset \mathbb{R}^3$  be a convex bounded domain with the boundary  $\partial\Omega \in C^3$  and specify time variable  $t \in [0, T]$ . Next, we supply the Cauchy problem by the appropriate b.c.
- In (85),  $\varepsilon(\mathbf{x}) = \varepsilon_r(\mathbf{x})\varepsilon_0$ ,  $\mu = \mu_r\mu_0$  are dielectric permittivity and permeability functions, respectively,  $\varepsilon_0, \mu_0$  are dielectric permittivity and permeability of free space, respectively.

Different CIPs for time-dependent equation for magnetic field (85) can be formulated.

## CIPs for magnetic wave propagation

 $\Omega$  $\mu_r(x) = ?$  $H(x, t) = g(x, t)$  on  $\partial\Omega$  $\Omega$  $\mu_r(x) = ?$   
 $\epsilon_r(x) = ?$  $H(x, t) = g(x, t)$  on  $\partial\Omega$ 

**Inverse Problem (MIP1)** Determine the relative magnetic permeability function  $\mu_r(x)$  in  $\Omega$  for  $x \in \Omega$  in nonconductive ( $\sigma(x) = 0$ ) media when the measured function  $g(x, t)$  s.t.

$$\mathbf{H}(x, t) = g(x, t), \forall (x, t) \in \partial\Omega \times (0, T].$$

is known in  $\Omega$ .

**Inverse Problem (MIP2)** Determine the functions  $\epsilon_r(x), \mu_r(x)$  in  $\Omega$  for  $x \in \Omega$  assuming that  $g(x, t)$  is known in  $\partial\Omega \times (0, T]$ .

# Maxwell's equations in 2D in a waveguide. TE and TM modes.

Let us assume that the structure being modeled extends to infinity in the  $z$ -direction with no change in the shape or position of its transverse cross section (case of a waveguide). If the incident wave is also uniform in the  $z$ -direction, then all partial derivatives of the fields with respect to  $z$  must equal zero. Under these conditions, the full set of Maxwell's curl equations given by (73) and (74) reduces to

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu} \left[ -\frac{\partial E_z}{\partial y} - (M_{source_x} + \sigma^* H_x) \right] \quad (86a)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu} \left[ \frac{\partial E_z}{\partial x} - (M_{source_y} + \sigma^* H_y) \right] \quad (86b)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu} \left[ \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - (M_{source_z} + \sigma^* H_z) \right] \quad (86c)$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon} \left[ \frac{\partial H_z}{\partial y} - (J_{source_x} + \sigma E_x) \right] \quad (87a)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\epsilon} \left[ -\frac{\partial H_z}{\partial x} - (J_{source_y} + \sigma E_y) \right] \quad (87b)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\epsilon} \left[ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - (J_{source_z} + \sigma E_z) \right] \quad (87c)$$

If we will group (86a), (86b), and (87c), which involve only  $H_x$ ,  $H_y$ , and  $E_z$  then we will set of field components to the *transverse-magnetic mode with respect to  $z$*  ( $TM_z$ ) in two dimensions.

If we will group (87a), (87b), and (86c), which involve only  $E_x$ ,  $E_y$ , and  $H_z$ . We shall designate this set of field components to the *transverse-electric mode with respect to  $z$*  ( $TE_z$ ) in two dimensions. >

# Maxwell's equations in 2D in a waveguide: TM mode

Recall: when we group (86a), (86b), and (87c), which involve only  $H_x$ ,  $H_y$ , and  $E_z$  then we will set of field components to the *transverse-magnetic mode with respect to z* (TM<sub>z</sub>) in two dimensions.

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu} \left[ -\frac{\partial E_z}{\partial y} - (M_{source_x} + \sigma^* H_x) \right] \quad (88a)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu} \left[ \frac{\partial E_z}{\partial x} - (M_{source_y} + \sigma^* H_y) \right] \quad (88b)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon} \left[ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - (J_{source_z} + \sigma E_z) \right] \quad (88c)$$

In non-conductive homogeneous isotropic media with  $M_{source_x} = M_{source_y} = J_{source_z} = 0$  the system above simplifies to

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu} \left[ -\frac{\partial E_z}{\partial y} \right] \quad (89a)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu} \left[ \frac{\partial E_z}{\partial x} \right] \quad (89b)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon} \left[ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] \quad (89c)$$

Assuming that for  $r = (x, y, z)$  waves propagates along the waveguide as

$$E_z(r, t) = \hat{E}_z(r, \omega) \cdot e^{-i\omega t}, H_x(r, t) = \hat{H}_x(r, \omega) \cdot e^{-i\omega t}, H_y(r, t) = \hat{H}_y(r, \omega) \cdot e^{-i\omega t} \quad (90)$$

## Maxwell's equations in 2D in a waveguide: TM mode

Applying it in the system (89) we get

$$-i\omega \hat{H}_x = \frac{1}{\mu} \left[ -\frac{\partial \hat{E}_z}{\partial y} \right] \quad (91a)$$

$$-i\omega \hat{H}_y = \frac{1}{\mu} \left[ \frac{\partial \hat{E}_z}{\partial x} \right] \quad (91b)$$

$$-i\omega \hat{E}_z = \frac{1}{\varepsilon} \left[ \frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} \right] \quad (91c)$$

From the first and second equations of system above we get

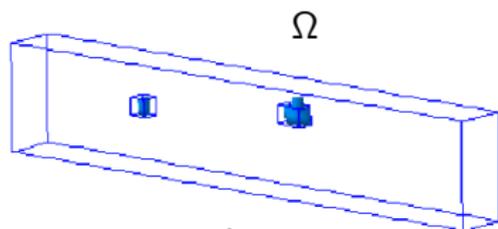
$$-i\omega \mu \frac{\hat{H}_x}{\partial y} = \left[ -\frac{\partial^2 \hat{E}_z}{\partial y^2} \right] \quad (92a)$$

$$-i\omega \mu \frac{\partial \hat{H}_y}{\partial x} = \left[ \frac{\partial^2 \hat{E}_z}{\partial x^2} \right] \quad (92b)$$

Then using (92a) - (92a) in (91c) we get the following equation:

$$\omega^2 \mu \varepsilon \hat{E}_z = -\frac{\partial^2 \hat{E}_z}{\partial x^2} - \frac{\partial^2 \hat{E}_z}{\partial y^2} \quad (93)$$

## CIPs for TM mode in a waveguide



$$\hat{E}_z(r, \omega) = g(r, \omega) \text{ on } \partial\Omega$$

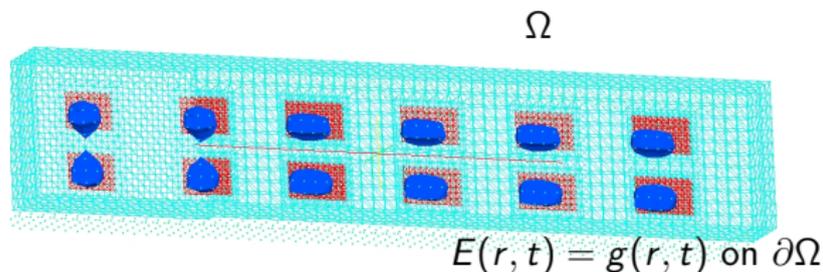
For the model problem

$$\omega^2 \mu \varepsilon \hat{E}_z = -\frac{\partial^2 \hat{E}_z}{\partial x^2} - \frac{\partial^2 \hat{E}_z}{\partial y^2} \quad (94)$$

we can formulate following CIP :

**Inverse Problem** Determine the dielectric permittivity function  $\varepsilon(r)$  in  $\Omega$  for known  $\omega$  and  $\mu$  for  $r = (x, y, z) \in \Omega$  in nonconductive ( $\sigma(r) = 0$ ) media when the measured function  $\hat{E}_z$  is known on  $\partial\Omega$ .

## CIPs for a waveguide



For the model problem

$$\varepsilon \frac{\partial^2 E}{\partial t^2} + \nabla \times (\mu^{-1} \nabla \times E) = 0, \text{ in } \Omega_T, \quad (95)$$

$$\nabla \cdot (\varepsilon E) = 0, \text{ in } \Omega_T, \quad (96)$$

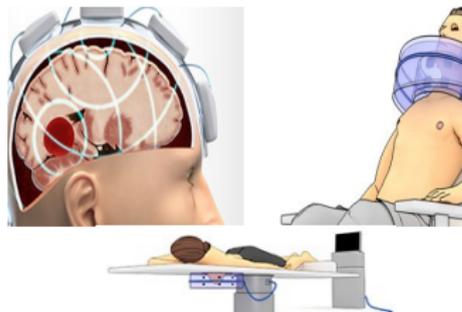
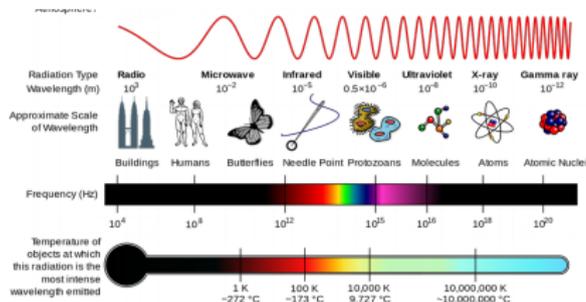
$$E(x, 0) = f_0(x), \quad E_t(x, 0) = f_1(x) \text{ in } \Omega, \quad (97)$$

$$E \times n = 0 \text{ on } \partial\Omega_T. \quad (98)$$

we can formulate following CIP :

**Inverse Problem** Determine the dielectric permittivity function  $\varepsilon(r)$  in  $\Omega$  for  $r = (x, y, z) \in \Omega$  in nonconductive ( $\sigma(r) = 0$ ) media when the measured function  $E(r, t)$  is known on  $\partial\Omega$ .

# Applications of CIPs for electric wave propagation



Left fig.: the electromagnetic spectrum (Wikipedia). Right fig.: Biomedical Microwave Imaging (frequencies around  $1 \text{ GHz} = 10^9 \text{ Hz}$ ) at the Department of Electrical Engineering at CTH, Chalmers, Göteborg, Sweden. Setup of Stroke Finder; microwave hyperthermia in cancer treatment and breast cancer detection, <https://www.chalmers.se/en/departments/e2/research/Signal-processing-and-Biomedical-engineering/>

<https://www.chalmers.se/en/departments/e2/research/Signal-processing-and-Biomedical-engineering/>



Detection of explosives and airport security (usually X-ray technique)

<https://www.rsdynamics.com/products/explosives-detectors/miniexplonix/>