Lecture Notes for the course "Numerical methods and machine learning algorithms for solution of inverse problems" MVE065/NFMV020 Chalmers/GU

Lecture 3: Microwave Imaging in monitoring of hyperthermia.
Analysis of solution of Helmholtz equation in 2D.

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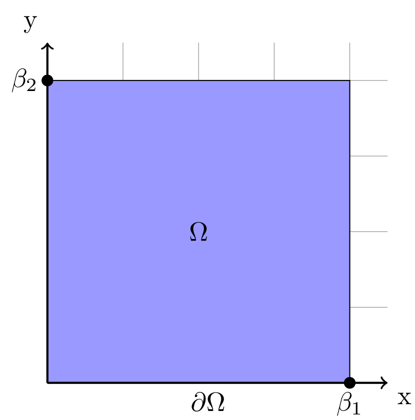
1 Introduction

This note presents analysis of solution of Helmholtz equation in two dimensions (2D) using finite difference discretization of Helmholtz equation. We are considering only the case of homogeneous boundary conditions.

First in Section 2 we discuss method of separation of variables to solve Helmholtz equation with homogeneous boundary conditions in 2D. Then Section 3 uses material of Section 2 for eigenvalue analysis of Helmholtz equation in 2D. In this section we also answer to the question: when the solution of Helmholtz equation is an ill-posed problem ?

We follow material of the classical book [2] for presentation of material in both sections. Convergence analysis for a finite difference approximation of the Dirichlet problem for the Helmholtz equation is presented in [3]. We refer to [1] for technique of derivation of error estimates for different PDE.



2 Method of separation of variables for Helmholtz equation in two dimensions

Let $\Omega \subset \mathbb{R}^2$ is a bounded simply connected space domain with boundary $\partial\Omega$ such that $\Omega := \{(x, y) : x \in [0, \beta_1], y \in [0, \beta_2]\}$, see Figure 1.

Our model equation is Helmholtz equation in two dimensions with homogeneous boundary conditions at $\partial\Omega$ and dielectric permittivity $\varepsilon(x, y) = 1$ in Ω:

$$\begin{aligned} \Delta u(x, y) + \omega^2 u(x, y) = 0, &\quad (x, y) \in \Omega, \\ u = 0 &\quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Δu in 2D is defined as:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{2}$$

In the method of separation of variables we will look a solution of (1) as product of two functions $X(x)$ and $Y(y)$:

$$u(x, y) = X(x)Y(y). \tag{3}$$

Our goal is to find solution of (1) in terms of functions $X(x)$ and $Y(y)$. To do that we substitute (3) into (1). First, we compute Δu in terms of $X(x)$ and $Y(y)$: after substitution of (3) into (2) we get:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 X(x)}{\partial x^2} Y(y); \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 Y(y)}{\partial y^2} X(x). \tag{4}$$

Next, substituting (4) into the Helmholtz equation in (1) we get :

$$\frac{\partial^2 X(x)}{\partial x^2} Y(y) + \frac{\partial^2 Y(y)}{\partial y^2} X(x) + \omega^2 X(x)Y(y) = 0. \tag{5}$$

Dividing both sides of (5) by $X(x)Y(y)$ we obtain:

$$\underbrace{\frac{\partial^2 X(x)}{\partial x^2}X(x)^{-1}}_{-\nu} + \underbrace{\frac{\partial^2 Y(y)}{\partial y^2}Y(y)^{-1}}_{-\mu} + \omega^2 = 0 \quad \Longleftrightarrow \tag{6a}$$

$$-(\nu+\mu)+\omega^2 = 0 \quad \Longleftrightarrow \tag{6b}$$

$$\lambda := \nu + \mu = \omega^2. \tag{6c}$$

Denoting by $X'' := \frac{\partial^2 X(x)}{\partial x^2}$ and $Y'' := \frac{\partial^2 Y(y)}{\partial y^2}$ we obtain two eigenvalue problems:

$$\begin{cases} \frac{X''}{X} = -\nu, \\ X(0) = 0, \quad X(\beta_1) = 0, \end{cases} \tag{7}$$

$$\begin{cases} \frac{Y''}{Y} = -\mu, \\ Y(0) = 0, \quad Y(\beta_2) = 0. \end{cases} \tag{8}$$

Boundary conditions in both eigenvalue problems (7), (8) are derived from the boundary conditions of Helmholtz equation (1). More precisely, we can rewrite (1) as

$$\begin{array}{lcl} \Delta u(x,y) + \omega^2 u(x,y) & = & 0 \\ u(0,y) & = & 0 \\ u(\beta_1,y) & = & 0 \\ u(x,0) & = & 0 \\ u(x,\beta_2) & = & 0. \end{array} \tag{9}$$

Here, β_1 and β_2 are sizes of the domain: Ω, see Figure 1. Using (9) we get the boundary conditions in eigenvalue problems (7), (8):

$$\begin{array}{llll} u(0,y) = X(0)Y(y) & = 0 \Longrightarrow X(0) & = 0 \text{ since } Y(y) \neq 0 \\ u(\beta_1,y) = X(\beta_1)Y(y) & = 0 \Longrightarrow X(\beta_1) & = 0 \text{ since } Y(y) \neq 0 \\ u(x,0) = X(x)Y(0) & = 0 \Longrightarrow Y(0) & = 0 \text{ since } X(x) \neq 0 \\ u(x,\beta_2) = X(x)Y(\beta_2) & = 0 \Longrightarrow Y(\beta_2) & = 0 \text{ since } X(x) \neq 0 \end{array} \tag{10}$$

In (10) functions $X(x), Y(y) \neq 0$ since we are looking for nontrivial solutions of (1). For problem (7) we seek the solution in the form $X(x) = Ce^{kx}$. Thus, $X''(x) = Ck^2e^{kx}$. Substituting expressions $X''(x) = Ck^2e^{kx}$ and $X(x) = Ce^{kx}$ into (7) we obtain

$$\begin{aligned} Ck^2e^{kx} + \nu Ce^{kx} = 0 \quad &\Longleftrightarrow \\ k^2 + \nu = 0 \quad &\Longleftrightarrow \\ k^2 = -\nu & \end{aligned} \tag{11}$$

Basing on the last equality in (11) we can consider three cases for ν:

1) When $\nu < 0$, then the problem (7) has solution given by

$$X(x) = C_1 e^{\sqrt{-\nu}x} + C_2 e^{-\sqrt{-\nu}x}. \tag{12}$$

Using boundary conditions of (7) in (12) we get

$$\begin{array}{rcl} X(0) = 0 & \Longrightarrow & X(0) = C_1 + C_2 = 0 \Longrightarrow C_1 = -C_2, \\ X(\beta_1) = 0 & \Longrightarrow & X(\beta_1) = C_1 e^{\sqrt{-\nu}\beta_1} + C_2 e^{-\sqrt{-\nu}\beta_1} = 0. \end{array} \tag{13}$$

Since $C_1 = -C_2$ then from last equation in (13) it follows that:

$$C_1(\underbrace{e^{\sqrt{-\nu}\beta_1} - e^{-\sqrt{-\nu}\beta_1}}_{\neq 0 \text{ since } \sqrt{-\nu} \text{ is real } (\nu<0)}) = 0. \tag{14}$$

Thus,

$$C_1 = 0 \implies C_1 = -C_2 = 0 \implies X(x) = 0. \tag{15}$$

We observe from the equation above that for $\nu < 0$ we obtain only trivial solution $X(x) = 0$.

2) When $\nu = 0$ we obtain again only trivial solution $X(x) = 0$.

3) When $\nu > 0$ the solution of (7) can be written as

$$X(x) = D_1 \cos(\sqrt{\nu}x) + D_2 \sin(\sqrt{\nu}x). \tag{16}$$

Using boundary condition in (7) we get

$$X(0) = D_1 \cos(0) + D_2 \sin(0) = 0 \implies D_1 = 0 \tag{17}$$

and

$$X(\beta_1) = \underbrace{D_1}_{=0} \cos(\sqrt{\nu}\beta_1) + D_2 \sin(\sqrt{\nu}\beta_1) = D_2 \sin(\sqrt{\nu}\beta_1). \tag{18}$$

Here, the constant $D_2 \neq 0$, meaning $\sin(\sqrt{\nu}\beta_1) = 0$. We observe that $\sin(\sqrt{\nu}\beta_1) = 0$ when $\sqrt{\nu} = \frac{\pi n}{\beta_1}, n \in \mathbb{Z}$. Thus, non-zero solution of (7) exists when

$$\nu = (\frac{\pi n}{\beta_1})^2, n \in \mathbb{Z}. \tag{19}$$

Taking the coefficient $D_2 = 1$, the solution of (7) will be written as $X_n(x) = \sin(\frac{\pi n}{\beta_1}x), n \in \mathbb{Z}$.

In the same way we can find non-trivial solution of (8). When $\mu > 0$ the solution of (8) can be written as

$$Y(y) = C_1 \cos(\sqrt{\mu}y) + C_2 \sin(\sqrt{\mu}y). \tag{20}$$

Using boundary condition in (8) we get $C_1 = 0$. We observe that $\sin(\sqrt{\mu}\beta_2) = 0$ when $\sqrt{\mu} = \frac{\pi m}{\beta_2}, m \in \mathbb{Z}$. Thus, the non-trivial solution of (8) will be when

$$\mu = (\frac{\pi m}{\beta_2})^2, \quad m \in \mathbb{Z}. \tag{21}$$

Again, taking coefficient $C_2 = 1$ the non-trivial solution of (8) will be:

$$Y_m(y) = \sin \frac{\pi m}{\beta_2} y, m \in \mathbb{Z}. \tag{22}$$

We can conclude that discrete solutions of the problem (1) can be written as

$$u_{nm}(x, y) = X_n(x)Y_m(y) = u_{nm} \sin \frac{\pi n}{\beta_1} x \sin \frac{\pi m}{\beta_2} y, \quad m, n \in \mathbb{Z} \tag{23}$$

and they are particular solutions for problem (1). Here, u_{nm} are coefficients which can be determined computationally. We observe that functions $u_{nm}(x, y)$ are orthogonal.

The common solution of (1) is given by summing particular solutions in (23):

$$u(x, y) = \sum_m \sum_n u_{nm}(x, y) = \sum_m \sum_n u_{nm} \sin \frac{\pi n}{\beta_1} x \sin \frac{\pi m}{\beta_2} y, \tag{24}$$

or as

$$u(x, y) = \sum_m \sum_n u_{nm} \rho_n(x) \rho_m(y) \tag{25}$$

with eigenfunctions

$$\begin{aligned} \rho_n(x) &= \sin \frac{\pi n}{\beta_1} x, \\ \rho_m(y) &= \sin \frac{\pi m}{\beta_2} y. \end{aligned} \tag{26}$$

Their corresponding eigenvalues are

$$\lambda_{nm} = \omega_{nm}^2 = v_n + \mu_m = (\frac{\pi n}{\beta_1})^2 + (\frac{\pi m}{\beta_2})^2. \tag{27}$$

3 Solvability of Helmholtz equation in 2D

Let us consider now the following problem for Helmholtz equation in 2D in Ω:

$$\begin{aligned} \Delta u(x, y) + \omega^2 \varepsilon(x, y) u(x, y) = f(x, y), \\ u(x, y) = 0 \ \ on \ \partial\Omega. \end{aligned} \tag{28}$$

3.1 Homogeneous medium

Let us analyze first the case $\varepsilon(x, y) = 1$ in (28). Thus, we consider the problem (1). As we already know from the previous section, we can seek solution of (1) in the form (25)

$$u(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{nm} \rho_n(x) \rho_m(y). \tag{29}$$

Substituting (29) in (28) noting that

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{nm} \rho_n(x) \rho_m(y), \tag{30}$$

we obtain

$$\begin{aligned} f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{nm} [\frac{\partial^2 \rho_n(x)}{\partial x^2} \rho_m(y) + \frac{\partial^2 \rho_m(y)}{\partial y^2} \rho_n(x)] \\ + \omega^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{nm} \rho_n(x) \rho_m(y). \end{aligned} \tag{31}$$

This equation can be rewritten as:

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}[\frac{\partial^2 \rho_n(x)}{\partial x^2}\rho_m(y) + \frac{\partial^2 \rho_m(y)}{\partial y^2}\rho_n(x) + \omega^2 \rho_n(x)\rho_m(y)]. \quad (32)$$

Here, $\rho_n(x)$ and $\rho_m(y)$ are eigenfunctions

$$\begin{aligned} \rho_n(x) &= \sin\frac{n\pi}{\beta_1}x, \quad n \in \mathbb{Z}, \\ \rho_m(y) &= \sin\frac{m\pi}{\beta_2}y, \quad m \in \mathbb{Z}, \end{aligned} \quad (33)$$

corresponding to the eigenvalues

$$\begin{aligned} \nu_n &= (\frac{n\pi}{\beta_1})^2, \\ \mu_n &= (\frac{m\pi}{\beta_2})^2, \end{aligned} \quad (34)$$

such that

$$\omega_{n,m}^2 = \lambda_{n,m} = \nu_n + \mu_n = (\frac{n\pi}{\beta_1})^2 + (\frac{m\pi}{\beta_2})^2. \quad (35)$$

Since $\rho_n(x) = \sin\frac{n\pi}{\beta_1}x$, then

$$\begin{aligned} \frac{\partial^2 \rho_n(x)}{\partial x^2} &= \frac{\partial}{\partial x}(\frac{\partial \rho_n(x)}{\partial x}) = -(\frac{n\pi}{\beta_1})^2 \sin\frac{n\pi}{\beta_1}x, \\ \frac{\partial \rho_n(x)}{\partial x} &= \frac{n\pi}{\beta_1}\cos\frac{n\pi}{\beta_1}x. \end{aligned} \quad (36)$$

Next, since $\rho_m(y) = \sin\frac{m\pi}{\beta_2}y$ we get:

$$\begin{aligned} \frac{\partial^2 \rho_m(y)}{\partial y^2} &= \frac{\partial}{\partial y}(\frac{\partial \rho_m(y)}{\partial y}) = -(\frac{m\pi}{\beta_2})^2 \sin\frac{m\pi}{\beta_2}y, \\ \frac{\partial \rho_m(y)}{\partial y} &= \frac{m\pi}{\beta_2}\cos\frac{m\pi}{\beta_2}y. \end{aligned} \quad (37)$$

Note that for $\Omega \subset \mathbb{R}^2, \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. Using (29) we get

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}\frac{\partial^2 \rho_n(x)}{\partial x^2}\rho_m(y) \\ \frac{\partial^2 u}{\partial y^2} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}\frac{\partial^2 \rho_m(y)}{\partial y^2}\rho_n(x) \end{aligned} \quad (38)$$

Using (36), (37) equations (38) can be rewritten as

$$\frac{\partial^2 u}{\partial x^2} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}(-\frac{n\pi}{\beta_1})^2(\sin\frac{n\pi}{\beta_1}x)(\sin\frac{m\pi}{\beta_2}y), \tag{39a}$$

$$\frac{\partial^2 u}{\partial y^2} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}(-\frac{m\pi}{\beta_2})^2(\sin\frac{m\pi}{\beta_2}y)(\sin\frac{n\pi}{\beta_1}x) \tag{39b}$$

Thus, for $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} =$ we get:

$$\begin{aligned}\Delta u = &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}\underbrace{(-\frac{n\pi}{\beta_1})^2}_{-\nu_n}\underbrace{(\sin\frac{n\pi}{\beta_1}x)}_{\rho_n(x)}\underbrace{(\sin\frac{m\pi}{\beta_2}y)}_{\rho_m(y)}\\ &+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}\underbrace{(-\frac{m\pi}{\beta_2})^2}_{-\mu_m}\underbrace{(\sin\frac{m\pi}{\beta_2}y)}_{\rho_m(y)}\underbrace{(\sin\frac{n\pi}{\beta_1}x)}_{\rho_n(x)},\end{aligned} \tag{40}$$

which we can write as

$$\Delta u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}[-\nu_n - \mu_m]\rho_n(x)\rho_m(y). \tag{41}$$

We also can rewrite the above equation as

$$\Delta u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}[-\underbrace{(\nu_n + \mu_m)}_{\lambda_{nm}}]\rho_n(x)\rho_m(y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}(-\lambda_{nm})\rho_n(x)\rho_m(y). \tag{42}$$

Substituting (42) into discretized Helmholtz equation (32) we have

$$\begin{aligned}f(x,y) = &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}(-\lambda_{nm})\rho_n(x)\rho_m(y)\\ &+\omega^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}\rho_n(x)\rho_m(y),\end{aligned} \tag{43}$$

or applying (30) in the left hand side of (43) we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{nm}\rho_n(x)\rho_m(y), = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-\lambda_{nm}+\omega^2)u_{nm}\rho_n(x)\rho_m(y). \tag{44}$$

From the above equation we observe that

$$u_{nm} = \frac{f_{nm}}{-\lambda_{nm}+\omega^2}. \tag{45}$$

Basing on (50) we can formulate conditions when the problem (1) is well-defined and when it is an ill-posed problem in 2D. These conditions are formulated in Lemma 1.

Lemma 1 The problem (1)

$$\Delta u(x,y) + \omega^2 u(x,y) = 0, \quad (x,y) \in \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

is well defined if $\omega^2 \neq \lambda_{nm}$ with

$$\lambda_{nm} = \nu_n + \mu_m = (\frac{n\pi}{\beta_1})^2 + (\frac{m\pi}{\beta_2})^2, \qquad n,m \in \mathbb{Z}. \tag{46}$$

More precisely, the problem (1)

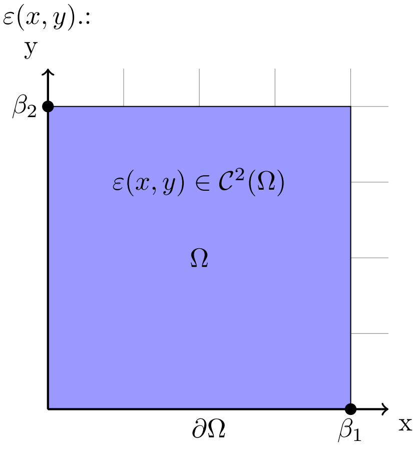
- 1) has unique solution if $\omega^2 \neq \lambda_{nm}, \forall m,n \in \mathbb{Z}$.
- 2) has no solution if $\omega^2 = \lambda_{nm}$ for some $m,n \in \mathbb{Z}$ and $f_{nm} \neq 0$.
- 3) has an infinite set of solutions if $\omega^2 = \lambda_{nm}$, for some $m,n \in \mathbb{Z}$ and $f_{nm} = 0$.

3.2 Non-homogeneous medium

Let us now analyze the case $\varepsilon(x,y) \in \mathcal{C}^2(\Omega)$ and consider the following problem for Helmholtz equation in 2D:

$$\begin{aligned} \Delta u(x,y) + \omega^2 \varepsilon(x,y) u(x,y) &= f(x,y), \quad (x,y) \in \Omega \\ u(x,y) &= 0 \quad on\ \partial\Omega. \end{aligned} \tag{47}$$

Figure below illustrates the computational domain with non-constant function $\varepsilon(x,y)$.:



Performing similar analysis as in section 3.1 we can obtain eigenvalues $\lambda := \nu + \mu$ and their discrete analog will be:

$$\lambda_{nm} = \nu_n + \mu_m = (\frac{\pi n}{\beta_1})^2 + (\frac{\pi m}{\beta_2})^2. \tag{48}$$

We approximate now $\varepsilon(x,y)$ by piecewise-constant functions ε_{nm} for all m,n such that equations (43) will be transformed to the following equations

$$
\begin{aligned}
f(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}(-\lambda_{nm})\rho_n(x)\rho_m(y) + \omega^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \varepsilon_{nm} u_{nm}\rho_n(x)\rho_m(y),\\
f(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-\lambda_{nm} + \varepsilon_{nm}\omega^2) u_{nm}\rho_n(x)\rho_m(y).
\end{aligned}
\tag{49}
$$

Applying (30) in the left hand side of (49) we get

$$
u_{nm} = \frac{f_{nm}}{-\lambda_{nm} + \varepsilon_{nm}\omega^2}. \tag{50}
$$

Thus, we can formulate following Lemma for the problem (47):
Lemma 2 The problem (49)

$$
\begin{aligned}
\Delta u(x,y) + \omega^2 \varepsilon u(x,y) = 0, &\quad (x,y) \in \Omega,\\
u = 0 &\quad \text{on } \partial\Omega,
\end{aligned}
$$

is well defined if $\omega^2 \varepsilon_{nm} \neq \lambda_{nm}$ with

$$
\lambda_{nm} = \nu_n + \mu_m = (\frac{n\pi}{\beta_1})^2 + (\frac{m\pi}{\beta_2})^2, \qquad n,m \in \mathbb{Z}. \tag{51}
$$

More precisely, the problem (49)

- 1) has unique solution if $\omega^2 \neq \frac{\lambda_{nm}}{\varepsilon_{nm}} \forall m,n \in \mathbb{Z}$.
- 2) has no solution if $\omega^2 = \frac{\lambda_{nm}}{\varepsilon_{nm}}$ for some $m,n \in \mathbb{Z}$ and $f_{nm} \neq 0$.
- 3) has an infinite set of solutions if $\omega^2 = \frac{\lambda_{nm}}{\varepsilon_{nm}}$, for some $m,n \in \mathbb{Z}$ and $f_{nm} = 0$.

3.3 Non-homogeneous medium in stabilized model

Let us now analyze the case $\varepsilon(x,y) \in \mathcal{C}^2(\Omega)$ for stabilized problem for Helmholtz equation in 2D:

$$
\begin{aligned}
\Delta u(x,y) + \omega^2 \varepsilon(x,y) u(x,y) + i\omega\alpha u = f(x,y), &\quad (x,y) \in \Omega\\
u(x,y) = 0 &\quad \textit{on } \partial\Omega.
\end{aligned}
\tag{52}
$$

Here, the term $i\omega\alpha u$ is a damping term with damping coefficient $\alpha > 0$ which plays roll of regularizaton of solution of Helmholtz equation.

Again, we perform analysis as in previous sections and obtain eigenvalues $\lambda := \nu + \mu$ and their discrete analog

$$\lambda_{nm} = \nu_n + \mu_m = (\frac{\pi n}{\beta_1})^2 + (\frac{\pi m}{\beta_2})^2. \tag{53}$$

We approximate now $\varepsilon(x, y)$ by piecewise-constant functions ε_{nm} for all m, n such that equations (52) will be transformed to the following regularized equations

$$\begin{aligned} f(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}(-\lambda_{nm})\rho_n(x)\rho_m(y) \\ &\quad + \omega^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \varepsilon_{nm} u_{nm}\rho_n(x)\rho_m(y) + i\omega\alpha \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{nm}\rho_n(x)\rho_m(y), \\ f(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (-\lambda_{nm} + \varepsilon_{nm}\omega^2 + i\omega\alpha) u_{nm}\rho_n(x)\rho_m(y). \end{aligned} \tag{54}$$

Applying (30) in the left hand side of (54) we get discrete solutions

$$u_{nm} = \frac{f_{nm}}{-\lambda_{nm} + \varepsilon_{nm}\omega^2 + i\omega\alpha}. \tag{55}$$

Thus, we can formulate following Lemma for stabilized problem (52):

Lemma 3 The regularized problem (52)

$$\begin{aligned} \Delta u(x,y) + \omega^2 \varepsilon u(x,y) + i\omega\alpha = 0, &\quad (x,y) \in \Omega, \\ u = 0 &\quad \text{on } \partial\Omega, \end{aligned}$$

is well defined for all ω.

Proof

Assume that denominator in (55) is zero, or

$$-\lambda_{nm} + \varepsilon_{nm}\omega^2 + i\omega\alpha = 0.$$

We want to find such $\omega > 0$ such that the above equation is true. Let us rewrite this equation in the form

$$\varepsilon_{nm}\omega^2 + i\alpha\omega - \lambda_{nm} = 0$$

and solve it for ω. We get following solutions of quadratic equation:

$$\omega = \frac{-i\alpha \pm \sqrt{D}}{2\varepsilon_{nm}} \tag{56}$$

with $D = 4\lambda_{nm}\varepsilon_{nm} - \alpha^2$. One can choose such $\alpha > 0$ that $D \geq 0$. From (56) it follows that ω is complex. However, ω is not complex, and thus,

$$-\lambda_{nm} + \varepsilon_{nm}\omega^2 + i\omega\alpha \neq 0.$$

References

[1] L. C. Evans, *Partial Differential Equations*, Amer. Math. Soc., Providence, RI, 1993.

[2] A. N. Tikhonov, A. A. Samarskii, Equations of Mathematical Physics (Dover Books on Physics) Reprint Edition

[3] S.A.Voitsekhovskii, V.L. Makarov, Yu. I. Rybak, Rate of convergence bounds of a finite difference approximation of the Dirichlet problem for the Helmholtz equation with a coefficient from the “negative” space, *Journal of Mathematical Sciences*, Vol.71, No.4, 1994.