LOGIC
University of Gothenburg

EXAM 2020–10–26
Logical theory part I, LOG111

Books and lecture notes are permitted, as are computers. However, no interaction with other people (except with the teacher) during the exam is allowed.

1. Deduce the following formulas when possible and prove that no such deduction exists otherwise: (4p)

   (a) $\exists x P(x,x) \to \exists x \exists y P(x,y)$

   (b) $\exists x \exists y P(x,y) \to \exists x P(x,x)$

   (c) $\forall x P(x) \to \forall x (P(x) \vee Q(x))$

   (d) $(\varphi \to (\psi \to \sigma)) \to ((\varphi \to \psi) \to \sigma)$

2. Let $\mathscr{L} = \emptyset$ be the empty language, i.e., there are no non-logical symbols (remember that equality is a logical symbol). Let $\Gamma = \{\forall x \forall y (x = y)\}$. (3p)

   (a) Show that for all $\mathscr{L}$-sentences φ either $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$.

   (b) Find a sentence φ in a larger language such that $\Gamma \nvdash \varphi$ and $\Gamma \nvdash \neg\varphi$.

3. For each pair of the following structures decide whether $M \equiv N$ and/or $M \cong N$. Motivate your answer. (3p)

   - $(\mathbb{N}, <)$,
   - $(\mathbb{N} \setminus \{0\}, <)$
   - $(\mathbb{Z}, <)$

   where $<$ is the usual strict order on natural numbers and integers.

4. Let Γ_1 and Γ_2 be sets of sentences in the same language $\mathscr{L}$ and suppose that an $\mathscr{L}$-structure M is a model of Γ_1 iff it is not a model of Γ_2. Show that both Γ_1 and Γ_2 are finitely axiomatizable. (Γ is finitely axiomatizable if there is a finite set of sentences Δ such that $\Gamma \vdash \varphi$ iff $\Delta \vdash \varphi$.) (3p)

CONTINUED ON NEXT PAGE.

5. Consider the following formal system for partial orders. Fix a non-empty (countable) set X. (12p)

- The symbols used are one constant symbol, c_x, for each $x \in X$, and two binary relation symbols: $<$ and $\not<$.
- The formulas are of the forms (and only of these forms) $c_x < c_y$, $c_x \not< c_y$, or $\perp$ for $x, y \in X$. Observe that there are no connectives and no quantifiers in this system, and thus no complex formulas.
- The structures are of the form (X, R) where X (same fixed set as above) is the domain and R is a partial order on X, i.e., a binary relation that is irreflexive and transitive.
- $\Gamma \vdash \varphi$ is defined by using the following rules:

$$\frac{c_x < c_x}{\perp}\,\mathrm{R} \qquad \frac{c_x < c_y \qquad c_y < c_z}{c_x < c_z}\,\mathrm{T} \qquad \frac{c_x < c_y \qquad c_x \not< c_y}{\perp}\,\perp \qquad \frac{\begin{array}{c}[c_x \not< c_y]\\ \vdots \\ \perp\end{array}}{c_x < c_y}\,\mathrm{RAA}_1 \qquad \frac{\begin{array}{c}[c_x < c_y]\\ \vdots \\ \perp\end{array}}{c_x \not< c_y}\,\mathrm{RAA}_2$$

(a) Give a deduction showing that $c_x < c_y \vdash c_y \not< c_x$.

(b) Give a definition of $(X, R) \models \varphi$ where R is a partial order on X and of $\Gamma \models \varphi$, where Γ is a set of formulas and φ is a single formula.

(c) Give an outline of a soundness proof for the formal system and prove the induction step for RAA_1 in detail.

(d) Prove that $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$.

(e) Let $\Gamma \vdash_i \varphi$ mean that φ is deducible from Γ without using RAA_1 or RAA_2. Prove the model existence lemma, i.e., that if $\Gamma \nvdash_i \perp$ then Γ is satisfiable.

(f) Prove that $\Gamma \models \varphi$ does not in general imply $\Gamma \vdash_i \varphi$.

Max points: 25. 12 points are required for Pass (G) and 18 for Pass with distinction (VG).

Fredrik Engström,
`fredrik.engstrom@gu.se`,
+46 – 31 – 786 6335